TABLE V
INVESTMENTS (MW)

	COOP	SQUO	PURPA
Utilities			
Combined cycle Natural gas	3000	3000	3000
Thermal coal	1588	2758	1588
Cogenerators			
Boiler coal 85% efficiency	0	6.421	0
Extraction turbine	0	576	0
Boiler + turbine conden. B	1000	1000	1000
Gas turbine	1738	0	1738
Exchange capacity	0	0	0

additional unit from cogenerator is its shadow price corrected by the 12% loss factor; for instance in summer day, this amounts to $82.17/(1 - 0.12) = 93.3$ mills, which is higher than the utility marginal cost.

Finally, it is worth noting that in the SQUO case, the cogenerators adopt an entirely different strategy in which they indeed buy electricity from the utility. Table V indicates that this strategy is also characterized by very different types of investments than in the COOP and PURPA cases.

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The Exact Slow–Fast Decomposition of the Algebraic Riccati Equation of Singularly Perturbed Systems

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Abstract—**The algebraic Riccati equation of singularly perturbed control systems is completely and exactly decomposed into two reduced-order algebraic Riccati equations corresponding to the slow and fast time scales. The pure-slow and pure-fast algebraic Riccati equations are nonsymmetric ones, but their $O(\epsilon)$ perturbations are symmetric. It is shown that the Newton method is very efficient for solving the obtained nonsymmetric algebraic Riccati equations. The presented method is very suitable for parallel computations. In addition, due to complete and exact decomposition of the Riccati equation, this procedure might produce a new insight in the two-time scale optimal filtering and control problems.**

I. Introduction

A linear singularly perturbed control system is given by

$$\begin{aligned} \dot{x}_1 &= A_1x_1 + A_2x_2 + B_1u \qquad x_1(t_0) = x_{10} \\ \epsilon\dot{x}_2 &= A_3x_1 + A_4x_2 + B_2u \qquad x_2(t_0) = x_{20} \end{aligned} \tag{1}$$

where $x_i \in R^{n_i}$, $i = 1, 2$, $u \in R^m$ are slow and fast state and control variables, respectively, and ϵ is a small positive parameter. As a parameter ϵ tends to zero, the solution behaves nonuniformly, producing a so-called stiff problem [1], [2].

The main idea of this note is to exploit the reduced-order slow and fast subsystems to find the exact solution of the global algebraic Riccati equation in terms of the reduced-order problems—both leading to the nonsymmetric algebraic Riccati equations: pure-slow and pure-fast. It is shown that the $O(\epsilon)$ perturbations of these nonsymmetric algebraic Riccati equations are symmetric ones and equal to the well-known first-order approximations of the slow and fast algebraic Riccati equations of singularly perturbed systems. The solutions of the symmetric reduced-order algebraic Riccati equations play the role of the initial guess for the Newton method which is very efficient for solving the obtained nonsymmetric Riccati equations. Due to complete and exact decomposition, the proposed method is very suitable for parallel computations.

II. Problem Formulation

With (1) consider the performance criterion

$$J = \frac{1}{2}\int_{t_0}^{\infty} \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T Q \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u^TRu \right\} dt \tag{2}$$

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with positive definite $\boldsymbol{R}$ and positive semidefinite $\boldsymbol{Q}$. The open-loop optimal control problem of (1), (2) has the solution

$$u(t) = -R^{-1}B^T p(t) \tag{3}$$

where $p(t) \in R^{n_1+n_2}$ is a costate variable satisfying [4]

$$\begin{bmatrix} \dot{x}(t) \\ \dot{p}(t) \end{bmatrix} = \begin{bmatrix} A & -S \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} \tag{4}$$

with

$$A = \begin{bmatrix} A_1 & A_2 \\ \frac{A_3}{\epsilon} & \frac{A_4}{\epsilon} \end{bmatrix}, \quad Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$B = \begin{bmatrix} B_1 \\ \frac{B_2}{\epsilon} \end{bmatrix}, \quad S = BR^{-1}B^T = \begin{bmatrix} S_1 & \frac{Z}{\epsilon} \\ \frac{Z^T}{\epsilon} & \frac{S_2}{\epsilon^2} \end{bmatrix}. \tag{5}$$

The optimal closed-loop control law has a very well-known form [4]

$$u = -R^{-1} \begin{bmatrix} B_1 \\ \frac{B_2}{\epsilon} \end{bmatrix}^T P \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -R^{-1}B^T Px \tag{6}$$

where P satisfies the algebraic Riccati equation given by

$$0 = PA + A^T P + Q - PSP. \tag{7}$$

Our main goal is to find the solution of (7) in terms of the solutions of reduced-order pure-slow and pure-fast algebraic Riccati equations.

III. Main Result

Partitioning p such that $p = [p_1^T \quad \epsilon p_2^T]^T$ with $p_1 \in R^{n_1}$ and $p_2 \in R^{n_2}$ and interchanging second and third rows in (4), we can get

$$\begin{bmatrix} \dot{x}_1 \\ \dot{p}_1 \\ \dot{x}_2 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} T_1 & T_2 \\ \frac{T_3}{\epsilon} & \frac{T_4}{\epsilon} \end{bmatrix} \begin{bmatrix} x_1 \\ p_1 \\ x_2 \\ p_2 \end{bmatrix} \tag{8}$$

where

$$T_1 = \begin{bmatrix} A_1 & -S_1 \\ -Q_1 & -A_1^T \end{bmatrix}, \quad T_2 = \begin{bmatrix} A_2 & -Z \\ -Q_2 & -A_3^T \end{bmatrix}$$

$$T_3 = \begin{bmatrix} A_3 & -Z^T \\ -Q_2^T & -A_2^T \end{bmatrix}, \quad T_4 = \begin{bmatrix} A_4 & -S_2 \\ -Q_3 & -A_4^T \end{bmatrix}. \tag{9}$$

It is important to notice that (8) retains the singular perturbation form. Also, the matrix T_4 is the Hamiltonian matrix of the fast subsystem, and it is nonsingular under stabilizability–detectability conditions imposed on the fast subsystem [4].

Introduce the notation

$$\begin{bmatrix} x_1 \\ p_1 \end{bmatrix} = w, \quad \begin{bmatrix} x_2 \\ p_2 \end{bmatrix} = \lambda \tag{10}$$

and the transformation [3] defined by

$$K^{-1} = \begin{bmatrix} I - \epsilon HL & -\epsilon H \\ L & I \end{bmatrix}, \quad K = \begin{bmatrix} I & \epsilon H \\ -L & I - \epsilon LH \end{bmatrix} \tag{11}$$

where L and H satisfy

$$T_4 L - T_3 - \epsilon L(T_1 - T_2 L) = 0 \tag{12}$$

$$-H(T_4 + \epsilon L T_2) + T_2 + \epsilon(T_1 - T_2 L)H = 0. \tag{13}$$

The unique conditions of (12) and (13) exist under the condition that T_4 is nonsingular [3].

The transformation (11) applied to (8) produces two completely decoupled subsystems

$$\dot{\eta} = (T_1 - T_2 L)\eta \tag{14}$$

and

$$\epsilon\dot{\xi} = (T_4 + \epsilon L T_2)\xi \tag{15}$$

where

$$\begin{bmatrix} \eta \\ \xi \end{bmatrix} = K^{-1} \begin{bmatrix} \omega \\ \lambda \end{bmatrix}. \tag{16}$$

The algebraic equations (12) and (13) can be solved by using any of the recursive algorithms developed in [5], [6].

The rearrangement and modification of variables in (8) is done by using the permutation matrix E_1 of the form

$$\begin{bmatrix} x_1 \\ p_1 \\ x_2 \\ p_2 \end{bmatrix} = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & I_{n_1} & 0 \\ 0 & I_{n_2} & 0 & 0 \\ 0 & 0 & 0 & \frac{I_{n_2}}{\epsilon} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ p_1 \\ \epsilon p_2 \end{bmatrix} = E_1 \begin{bmatrix} x \\ p \end{bmatrix}. \tag{17}$$

Combining (16) and (17), we obtain the relationship between the original coordinates and the new ones

$$\begin{bmatrix} \eta_1 \\ \xi_1 \\ \eta_2 \\ \xi_2 \end{bmatrix} = E_2^T K^{-1} E_1 \begin{bmatrix} x \\ p \end{bmatrix} = \Pi \begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_3 & \Pi_4 \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} \tag{18}$$

where E_2 is a permutation matrix in the form

$$E_2 = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & I_{n_1} & 0 \\ 0 & I_{n_2} & 0 & 0 \\ 0 & 0 & 0 & I_{n_2} \end{bmatrix}. \tag{19}$$

Since $p = Px$, where P satisfies the algebraic Riccati equation (7), it follows that

$$\begin{bmatrix} \eta_1 \\ \xi_1 \end{bmatrix} = (\Pi_1 + \Pi_2 P)x, \quad \begin{bmatrix} \eta_2 \\ \xi_2 \end{bmatrix} = (\Pi_3 + \Pi_4 P)x. \tag{20}$$

In the original coordinates, the required optimal solution has a closed-loop nature. We have the same attribute for the new systems (14) and (15); that is

$$\begin{bmatrix} \eta_2 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \xi_1 \end{bmatrix}. \tag{21}$$

Then, (20) and (21) yield

$$\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} = (\Pi_3 + \Pi_4 P)(\Pi_1 + \Pi_2 P)^{-1}. \tag{22}$$

Following the same logic, we can find P reversely by introducing

$$E_1^{-1} K E_2 = \Omega = \begin{bmatrix} \Omega_1 & \Omega_2 \\ \Omega_3 & \Omega_4 \end{bmatrix} \tag{23}$$

where

$$E_1^{-1} = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & I_{n_2} & 0 \\ 0 & I_{n_1} & 0 & 0 \\ 0 & 0 & 0 & \epsilon I_{n_2} \end{bmatrix} \tag{24}$$

and it yields

$$P = \left(\Omega_3 + \Omega_4 \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}\right)\left(\Omega_1 + \Omega_2 \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}\right)^{-1}. \tag{25}$$

It is shown in Appendix I that required matrices in (22) and (25) are invertible for sufficiently small ϵ. Partitioning (14) and (15) as

$$\begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = (T_1 - T_2 L)\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \tag{26}$$

$$\epsilon\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = (T_4 + \epsilon L T_2)\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \tag{27}$$

and using (21) yield to two reduced-order nonsymmetric algebraic Riccati equations

$$0 = P_1 a_1 - a_4 P_1 - a_3 + P_1 a_2 P_1 \tag{28}$$

$$0 = P_2 b_1 - b_4 P_2 - b_3 + P_2 b_2 P_2 \tag{29}$$

where

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = \begin{bmatrix} A_1 - A_2 L_1 + Z L_3 & -S_1 - A_2 L_2 + Z L_4 \\ -Q_1 + Q_2 L_1 + A_3^T L_3 & -A_1^T + Q_2 L_2 + A_3^T L_4 \end{bmatrix}$$

$$\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = \begin{bmatrix} A_4 + \epsilon(L_1 A_2 - L_2 Q_2) & -S_2 - \epsilon(L_1 Z + L_2 A_3^T) \\ -Q_3 + \epsilon(L_3 A_2 - L_4 Q_2) & -A_4^T - \epsilon(L_3 Z + L_4 A_3^T) \end{bmatrix} \tag{30}$$

with

$$L = \begin{bmatrix} L_1 & L_2 \\ L_3 & L_4 \end{bmatrix}. \tag{31}$$

The pure-slow algebraic Riccati equation (28) is nonsymmetric and it is given by

$$P_1(A_1 - A_2 L_1 + Z L_3) + (A_1 - L_2^T Q_2^T - L_4^T A_3)^T P_1 + (Q_1 - Q_2 L_1 - A_3^T L_3) - P_1(S_1 + A_2 L_2 - Z L_4) P_1 = 0. \tag{32}$$

The pure-fast algebraic Riccati equation (29) is also nonsymmetric

$$P_2(A_4 + \epsilon(L_1 A_2 - L_2 Q_2)) + (A_4^T + \epsilon(L_3 Z + L_4 A_3^T))P_2 + (Q_3 - \epsilon(L_2 A_2 - L_4 Q_2)) - P_2(S_2 + \epsilon(L_1 Z + L_2 A_3^T))P_2 = 0 \tag{33}$$

but its $O(\epsilon)$ approximation is a symmetric one, that is

$$P_2 A_4 + A_4^T P_2 + Q_3 - P_2 S_2 P_2 + O(\epsilon) = 0. \tag{34}$$

In addition, it can be shown (see Appendix II) that (32) is an $O(\epsilon)$ perturbation of the first-order approximate slow algebraic Riccati equation obtained in [10]

$$P_s A_s + A_s P_s + Q_s - P_s S_s P_s = 0 \tag{35}$$

where A_s, Q_s, and S_s can be found in [10].

The nonsymmetric algebraic Riccati equation was studied in [7]. An algorithm for solving a general nonsymmetric algebraic Riccati equation was derived in [8].

Using (34), (35) and the implicit function theorem [11], the existence of the unique solutions of (32) and (33) is guaranteed by the following lemma.

Lemma: If the triples $(A_4, B_2, \sqrt{Q_3})$ and $(A_s, \sqrt{S_s}, \sqrt{Q_s})$ are stabilizable–detectable, then $\exists \epsilon_0 > 0$ s.t. $\forall \epsilon \le \epsilon_0$ unique solutions of (32) and (33) exist.

From (34) one can obtain an $O(\epsilon)$ approximation for P_2 as

$$P_2^{(0)} A_4 + A_4^T P_2^{(0)} + Q_3 - P_2^{(0)} S_2 P_2^{(0)} = 0. \tag{36}$$

Having obtained a good initial guess, the Newton-type algorithm can be used very efficiently for solving (34). The Newton algorithm is given by

$$P_2^{(i+1)}(b_1 - b_2 P_2^{(i)}) - (b_4 - P_2^{(i)} b_2) P_2^{(i+1)} = b_3 + P_2^{(i)} b_2 P_2^{(i)} \qquad i = 0, 1, 2, \cdots \tag{37}$$

with an initial guess obtained from (36).

The pure-slow equation (32) can be solved by using the Newton algorithm also, with an initial guess obtained from (35). The Newton algorithm for (32) is given by

$$P_1^{(i+1)}(a_1 - a_2 P_1^{(i+1)}) - (a_4 - P_1^{(i)} a_2) P_1^{(i+1)} = a_3 + P_1^{(i)} a_2 P_1^{(i)} \qquad P_1^{(0)} = P_s, \qquad i = 0, 1, 2, \cdots. \tag{38}$$

It is important to notice that the total number of scalar quadratic algebraic equations in (32) and (33) is $n_1^2 + n_2^2$. On the other hand, the global algebraic Riccati equation (6) contains $(1/2)\,(n_1 + n_2)(n_1 + n_2 + 1)$ scalar equations. Thus, the proposed method can reduce the number of equations if

$$n_1^2 + n_2^2 < \tfrac{1}{2}(n_1 + n_2)(n_1 + n_2 + 1) \tag{39}$$

or

$$(n_1 - n_2)^2 < n_1 + n_2. \tag{40}$$

Using solutions of both pure-slow and pure-fast Riccati equations and formulas (21) and (26), we can get completely decoupled slow and fast subsystems in the form

$$\dot{\eta}_1 = (a_1 + a_2 P_1)\eta_1$$

$$\epsilon\dot{\xi}_1 = (b_1 + b_2 P_2)\xi_1. \tag{41}$$

The global solution in the original coordinates is then obtained at any time instant by using the formula (20), that is

$$x = (\Pi_1 + \Pi_2 P)^{-1}\begin{bmatrix} \eta_1 \\ \xi_1 \end{bmatrix} \tag{42}$$

where P is given by (25).

A numerical example that demonstrates the efficiency of the proposed method can be found in [12].

IV. Conclusion

In summary, we have obtained the solution of the global (full-order) algebraic Riccati equation of singularly perturbed systems in terms of pure-slow and pure-fast reduced-order algebraic Riccati equations. Instead of solving $(n_1 + n_2)(n_1 + n_2 + 1)/2$ equations for symmetric P in (7), we solve $(n_1^2 + n_2^2)$ equations in (32) and (33). This is more efficient if n_1, n_2 are selected to be close to each other. Furthermore, due to the split into two independent subsystems, the advantage of the parallel computation becomes significant in this case.

Appendix I

It is easy to show that

$$\begin{bmatrix} \Omega_1 & \Omega_2 \\ \Omega_3 & \Omega_4 \end{bmatrix} = E_1^{-1} K E_2$$

$$= \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ -L_1 & I_{n_2} & -L_2 & 0 \\ 0 & 0 & I_{n_1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + O(\epsilon) \quad \text{(A1)}$$

which implies

$$\Omega_1 = \begin{bmatrix} I_{n_1} & 0 \\ -L_1 & I_{n_2} \end{bmatrix} + O(\epsilon), \qquad \Omega_2 = \begin{bmatrix} 0 & 0 \\ -L_2 & 0 \end{bmatrix} + O(\epsilon). \quad \text{(A2)}$$

Then, the matrix

$$\Omega_1 + \Omega_2 \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} = \begin{bmatrix} I_{n_1} & 0 \\ -L_1 - L_2 P_1 & I_{n_2} \end{bmatrix} + O(\epsilon) \quad \text{(A3)}$$

is invertible for sufficiently small values of ϵ.

Similarly

$$\begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_3 & \Pi_4 \end{bmatrix} = E_2^T K^{-1} E_1$$

$$= \begin{bmatrix} I_{n_1} & 0 & 0 & -H_2 \\ L_1 & I_{n_2} & 0 & 0 \\ 0 & 0 & I_{n_1} & -H_4 \\ L_3 & 0 & 0 & \frac{I_{n_2}}{\epsilon} \end{bmatrix} + O(\epsilon) \quad \text{(A4)}$$

with

$$\Pi_1 = \begin{bmatrix} I_{n_1} & 0 \\ L_1 & I_{n_2} \end{bmatrix} + O(\epsilon), \qquad \Pi_2 = \begin{bmatrix} 0 & -H_2 \\ 0 & 0 \end{bmatrix} + O(\epsilon) \quad \text{(A5)}$$

imply that the matrix

$$\Pi_1 + \Pi_2 P = \begin{bmatrix} I_{n_1} & 0 \\ L_1 & I_{n_2} \end{bmatrix} + O(\epsilon) \quad \text{(A6)}$$

is invertible for sufficiently small values of ϵ. In this Appendix, we have used the following notation for the partitioned matrix H:

$$H = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}. \quad \text{(A7)}$$

Appendix II

From (A1) we have

$$\Omega_3 + \Omega_4 \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} = \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix} + O(\epsilon). \quad \text{(B1)}$$

Using (B1) and (A3) in formula (25) produces

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_{n_1} & 0 \\ -L_1 - L_2 P_1 & I_{n_2} \end{bmatrix}^{-1} + O(\epsilon) \quad \text{(B2)}$$

or

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix} + O(\epsilon). \quad \text{(B3)}$$

It is very well known that the structure of the solution of P is given by [1], [2]

$$P = \begin{bmatrix} P_0 & \epsilon P_m \\ \epsilon P_m^T & \epsilon P_f \end{bmatrix} \quad \text{(B4)}$$

which implies

$$P_1 = P_0 + O(\epsilon). \quad \text{(B5)}$$

On the other hand, P_0 is $O(\epsilon)$ close to the solution of (35), that is to P_s [10] so that

$$P_1 = P_s + O(\epsilon). \quad \text{(B6)}$$

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