different controllable subspaces, we ask if arbitrary rational inputs do not generate an even larger controllable subspace. (Of course, in this case, one may not hope to have smooth trajectories.)

Definition 4.1: A regular system (E, A, B) will be said to be *almost controllable* if, given any initial condition $Ex(0_-)$, there exists a finite time $T > 0$, a $U(s) \in \mathbf{R}^n(s)$, and a unique $X(s) \in \mathbf{R}^n(s)$ so that $(sE - A) - BU(s) = Ex(0_-)$ and $x(T) = 0$. A regularizable system will be said to be "*almost controllable*" if (E_K, A_F, B) is almost controllable for some F and K.

Theorem 4.3: A regularizable system (E, A, B) is "almost controllable" iff $\mathbf{R}_a^* = \mathbf{R}^n$.

Proof: Given a regularizable system (E, A, B), we first choose a K so that: 1) $\mathbf{Ker} E_K \cap V^* = 0$ and ii) $\mathbf{Im} B \subset E_K V^*$. Note that regularizability of (E, A, B), Theorem 3.2, and Lemma 2.2 guarantee the existence of such a K. Then, (E_K, A, B) is regular with V^* as its initial manifold and therefore, there exist two nonsingular matrices W and V so that the Weierstrass form $(W^{-1}E_K V, W^{-1}AV, W^{-1}B)$ [1] of (E_K, A, B) becomes [5]

$$\dot{x}_1 = J\dot{x}_1 + B_1 u$$

$$N\dot{x}_2 = x_2$$

where N is a Jordan-form matrix corresponding to the zero eigenvalue. Clearly, (E, A, B) is "almost controllable" iff (E_K, A, B) is "almost controllable" and, as it is regular, (E_K, A, B) is "almost controllable" iff it is almost controllable, or equivalently iff $(W^{-1}E_K V, W^{-1}AV, W^{-1}B)$ is almost controllable. As $x_2(t) = 0$ for $t > 0$, almost controllability of $(W^{-1}E_K V, W^{-1}AV, W^{-1}B)$ is equivalent to the controllability of the state-space system (J, B_1). However, (J, B_1) is controllable iff $\mathbf{R}_a^*(W^{-1}E_K V, W^{-1}AV, W^{-1}B) = \mathbf{R}^n$ [6]. As $\mathbf{R}_a^*(W^{-1}E_K V, W^{-1}AV, W^{-1}B) = \mathbf{R}_a^*(E_K, A, B) = \mathbf{R}_a^*(E, A, B)$ the result follows. Q.E.D.

It should now be clear that the property of controllability depends on the type of feedback to be applied because, in general, it is quite sensitive to the inputs used to drive the given initial condition to the origin. On the other hand, as suggested by its noninvariance with respect to the type of feedback to be admitted, the property of reachability is independent of the class of inputs to be used to reach a given point. Or, more precisely:

Lemma 4.2: Let (E, A, B) be regularizable and let $y \in \mathbf{R}^n$ be given. If there exist $T > 0$, $X(s) \in \mathbf{R}^n(s)$ and $U(s) \in \mathbf{R}^n(s)$ satisfying:

a) $(sE - A) - BU(s) = 0$,

b) $x(T) = y$ [where, as before, $x(t)$ denotes the inverse Laplace transform of $X(s)$], then there exist $X_1(s) \in \mathbf{R}_{sp}^n(s)$ and $U_1(s) \in \mathbf{R}_{sp}^n(s)$ enjoying the same properties a) and b).

This result demonstrates that if a point y can be reached from the origin in finite time along an impulsive trajectory generated by an impulsive input, then the same point can be reached from the origin in finite time along a smooth trajectory generated by a smooth input. This explains why enlarging the class of inputs from $\mathbf{R}_{sp}^n(s)$ to $\mathbf{R}_p^n(s)$ (or, equivalently, introducing derivative feedback) does not enlarge the reachable space of the system; and why there is no need to introduce the concept of "almost reachability."

Proof (Lemma 4.2): First, choose an F so that (E, A_F, B) is regular. Let y, T, $X(s)$, and $U(s)$ be as in the statement of the lemma. Expand $X(s)$ and $U(s)$ as in (4.2) and (4.3). It is easy to show that $x_k \in \mathbf{R}_a^*$ for all $k \leq 0$ and $x_k \in V^*$ for all $k > 0$. As the initial condition is zero, we also have $Ex_1 = Ax_0 + Bu_0$. Then, $x_0 \in \mathbf{R}_a^* \cap A^{-1}(EV^* + \mathbf{Im} B) = \mathbf{R}_a^* \cap V^* = \mathbf{R}^*$ [6], and $x_1 \in V^* \cap E^{-1}(A\mathbf{R}_a^* + \mathbf{Im} B) = \mathbf{R}^*$. Then, an easy induction argument shows that $x_k \in \mathbf{R}^*$ for all k. As $x(t) \in Sp\{\cdots, x_{-1}, x_0, x_1, x_2, \cdots,\} \subset \mathbf{R}^*$, it follows that $y = x(T)$ is in $\mathbf{R}^*$. As (E, A_F, B) is regular and as $y \in \mathbf{R}^*$, there exists a smooth input $u(t)$ with a strictly proper Laplace transform $U(s)$ so that the trajectory $x(t)$ is smooth with strictly proper Laplace transform $X(s)$ which satisfies $x(t) = y$ [5]. Then, taking $X_1(s) = X(s)$ and $U_1(s) = U(s) + FX(s)$ completes the proof. Q.E.D.

V. Conclusions

Arguing against the assumption of regularity which overwhelms the literature on continuous-time singular systems, we have introduced the notion of regularizability and have shown that, unlike regularity, it is invariant under linear "state" feedback. We have established that a modification of the definition of reachability so as to make it dependent on regularizability rather than regularity is not only possible but also desirable as it is invariant under linear "state" feedback. It has also been shown that a similar remedy of the noninvariance of controllability under linear feedback turns out to be somewhat involved in the sense that the correct way to define controllability depends on the type of feedback law to be used. Thus, we have defined "controllability by proportional feedback," "controllability by derivative feedback," and "controllability by proportional-plus-derivative feedback" and have shown that the last two are the one and the same property. Apart from their feedback and geometric characterizations, dynamical interpretations of these concepts have also been introduced. It has been shown that, under the quite natural assumption that $\mathbf{Im} E + \mathbf{Im} A + \mathbf{Im} B = \mathbf{R}^n$, regularizability is the condition whereby given any initial condition one can find at least one admissible input which generates a trajectory. The dependence of the definition of controllability on the type of feedback has been reinterpreted as a symptom of its dependence on the type of inputs to be used to drive the given initial condition to the origin. It has been established that the use of derivative feedback in the closed-loop system is equivalent to using an open-loop control which has a Dirac delta term. Finally, the definition of reachability has been shown to be insensitive not only to the changes in the type of feedback inputs, but also to possible changes in the class of open-loop inputs.

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Optimal Reduced-Order Solution of the Weakly Coupled Discrete Riccati Equation

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***Abstract*—The optimal solution of the weakly coupled algebraic discrete Riccati equation is obtained in terms of a reduced-order continuous**

Manuscript received January 9, 1989; revised July 28, 1989 and October 13, 1989.

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IEEE Log Number 9036819.

type algebraic Riccati equation via the use of a bilinear transformation. The proposed method has a rate of convergence of $O(\epsilon^2)$ where ϵ represents a small coupling parameter. The method is applicable under mild assumptions.

I. Introduction

Linear weakly coupled continuous systems have been studied in [1]–[12]. However, linear weakly coupled discrete systems have not been studied due to the fact that the partitioned form of the main equation of the optimal linear control theory, Riccati equation, has a very complicated form in the discrete-time domain. This note overcomes that problem by the use of a bilinear transformation, which is applicable under mild assumptions, such that the solution of the discrete algebraic Riccati equation of weakly coupled systems is obtained using results from the corresponding continuous-time equation.

The algebraic Riccati equation of weakly coupled linear discrete systems is given by

$$P = A^T PA + Q - A^T PB(B^T PB + R)^{-1} B^T PA, \quad R > 0, Q \geq 0 \tag{1}$$

where

$$A = \begin{pmatrix} A_1 & \epsilon A_2 \\ \epsilon A_3 & A_4 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & \epsilon B_2 \\ \epsilon B_3 & B_4 \end{pmatrix},$$

$$Q = \begin{pmatrix} Q_1 & \epsilon Q_2 \\ \epsilon Q_2^T & Q_3 \end{pmatrix}, \quad R = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix},$$

and ϵ is a small coupling parameter. Due to the block dominant structure of the problem matrices, the required solution P has the form

$$P = \begin{pmatrix} P_1 & \epsilon P_2 \\ \epsilon P_2^T & P_3 \end{pmatrix}. \tag{2}$$

The main goal in weakly coupled system theory is to obtain the required solution in terms of reduced-order problems, namely, subsystems. In the case of the weakly coupled algebraic discrete Riccati equation, the inversion of the partitioned matrix $(B^T PB + R)$ will produce a lot of nonzero terms and make the corresponding approach computationally very involved, even though one is faced with reduced-order numerical problems.

To solve this problem, we have used a bilinear transformation introduced in [13] to transform the discrete Riccati equation (1) into a continuous-time algebraic Riccati equation of the following form:

$$A_c^T P_c + P_c A_c + Q_c - P_c S_c P_c = 0, \qquad S_c = B_c R_c^{-1} B_c^T. \tag{3}$$

The solution of (1) is equal to the solution of (3). Appendix I shows that (3) preserves the structure of weakly coupled systems and can be efficiently solved in terms of the reduced-order problems using the fixed-point type method developed in [7]. The required solution is then obtained with the rate of convergence of $O(\epsilon^2)$.

II. Computational Algorithm

Since the proposed algorithm for a discrete algebraic Riccati equation combines features of the bilinear transformation [13] and the fixed-point algorithm developed in [7] for the weakly coupled continuous algebraic Riccati equation, we will briefly summarize the main results obtained in [13] and [7].

The bilinear transformation states that (1) and (3) have the same solution if the following relations hold [13];

$$A_c = I - 2D^{-T} \tag{4a}$$

$$S_c = 2(I + A)^{-1} S_d D^{-1}, \qquad S_d = BR^{-1}B^T \tag{4b}$$

$$Q_c = 2D^{-1}Q(I + A)^{-1} \tag{4c}$$

$$D = (I + A^T) + Q(I + A)^{-1} S_d \tag{4d}$$

and assuming that $(I + A)^{-1}$ exists. Matrix D has been shown to be invertible [15]. The physical interpretation of the transformation between the continuous and discrete type algebraic Riccati equation is discussed in [13].

The proposed algorithm will be valid under the following assumption.

Assumption 1: The system matrix A has no eigenvalues located at -1.

It is important to point out that the eigenvalues located in the neighborhood of -1 will produce ill conditioning with respect to matrix inversion and make the algorithm numerically unstable.

It can be verified that the weakly coupled structure of matrices defined in (1) will produce the weakly coupled structure of transformed matrices given in (4) (Appendix I). The compatible partitions of these matrices are

$$A_c = \begin{pmatrix} A_{11} & \epsilon A_{12} \\ \epsilon A_{21} & A_{22} \end{pmatrix}, \quad S_c = \begin{pmatrix} S_{11} & \epsilon S_{12} \\ \epsilon S_{12}^T & S_{22} \end{pmatrix}$$

$$Q_c = \begin{pmatrix} Q_{11} & \epsilon Q_{12} \\ \epsilon Q_{12}^T & Q_{22} \end{pmatrix}, \quad P_c = P = \begin{pmatrix} P_1 & \epsilon P_2 \\ \epsilon P_2^T & P_3 \end{pmatrix}.$$

These partitions have to be performed by a computer only in the process of calculations and there is no need for the corresponding analytical expressions.

Solution of (3) can be found in terms of the reduced-order problems by imposing standard stabilizability–detectability assumptions on the subsystems. The efficient fixed-point reduced-order algorithm for solving (3) is obtained in [7].

The $O(\epsilon^2)$ approximation of (3) is obtained from the following decoupled set of equations:

$$\boldsymbol{P}_1 A_{11} + A_{11}^T \boldsymbol{P}_1 + Q_{11} - \boldsymbol{P}_1 S_{11} \boldsymbol{P}_1 = 0 \tag{5a}$$

$$\boldsymbol{P}_3 A_{22} + A_{22}^T \boldsymbol{P}_3 + Q_{22} - \boldsymbol{P}_3 S_{22} \boldsymbol{P}_3 = 0 \tag{5b}$$

$$\boldsymbol{P}_2(A_{22} - S_{22}\boldsymbol{P}_3) + (A_{11} - S_{11}\boldsymbol{P}_1)^T \boldsymbol{P}_2 + \boldsymbol{P}_1 A_{12} + A_{21}^T \boldsymbol{P}_3 + Q_{12} - \boldsymbol{P}_1 S_{12} \boldsymbol{P}_3 = 0. \tag{5c}$$

Unique positive semidefinite stabilizing solutions of (5a) and (5b) exist under the following assumption.

Assumption 2: Triples $(A_{ii}, \sqrt{S_{ii}}, \sqrt{Q_{ii}})$, $i = 1, 2$ are stabilizable-detectable.

Defining the approximation errors as

$$P_i = \boldsymbol{P}_i + \epsilon^2 E_i, \qquad i = 1, 2, 3 \tag{6}$$

the fixed-point type algorithm, with the rate of convergence of $O(\epsilon^2)$, is obtained in [7] in the decoupled form as

$$E_1^{(j+1)} \Delta_1 + \Delta_1^T E_1^{(j+1)} = M_1^{(j)} \tag{7a}$$

$$E_3^{(j+1)} \Delta_2 + \Delta_2^T E_3^{(j+1)} = M_3^{(j)} \tag{7b}$$

$$E_2^{(j+1)} \Delta_2 + \Delta_1^T E_2^{(j+1)} + E_1^{(j+1)} \Delta_{12} + \Delta_{21}^T E_3^{(j+1)} = M_2^{(j,j+1)} \tag{7c}$$

with $j = 0, 1, 2, \cdots$, and $E_1^{(0)} = 0$, $E_2^{(0)} = 0$, $E_3^{(0)} = 0$ where newly defined matrices are given in Appendix II. Note that Δ_1 and Δ_2 are stable matrices [1].

The rate of convergence of (7) is $O(\epsilon^2)$ [7], that is

$$\|P_i - P_i^{(j)}\| = O(\epsilon^{2j}), \qquad i = 1, 2, 3; \quad j = 0, 1, 2, \cdots, \tag{8}$$

where

$$P_i^{(j)} = \boldsymbol{P}_i + \epsilon^2 E_i^{(j)}, \qquad i = 1, 2, 3; \quad j = 0, 1, 2, \cdots, . \tag{9}$$

The proposed algorithm for the reduced-order solution of the discrete algebraic Riccati equation under conditions stated in Assumptions 1 and 2 has the following form.

1) Transform (1) into (3) using (4).
2) Solve (3) using the reduced-order algorithm (5)–(7).

TABLE I
REDUCED-ORDER SOLUTION OF THE DISCRETE WEAKLY COUPLED ALGEBRAIC RICCATI EQUATION

j	$P_1^{(j)}$	$P_2^{(j)}$	$P_3^{(j)}$
0	39.937 2.6157 1.5479	3.5566 2.5105 30.533 0.7706 0.4539 3.4863	1.4838 0.3050 2.1505 1.2121 1.8158 26.135
1	51.477 3.7414 1.6922	4.4048 2.9652 34.448 0.8767 0.5108 3.9782	1.5641 0.3488 2.5341 1.2353 2.0101 27.622
2	56.881 4.3019 1.7581	4.7920 3.1561 35.815 0.9224 0.5340 0.4155	1.5985 0.3665 2.6762 1.2441 2.0740 27.970
4	60.175 4.6541 1.7983	5.0253 3.2657 36.500 0.9492 0.5478 4.2398	1.6186 0.3763 2.7440 1.2487 2.1036 28.116
6	60.733 4.7144 1.8050	5.0644 3.2836 36.600 0.9535 0.5488 4.2514	1.6218 0.3778 2.7529 1.2494 2.1076 28.136
8	60.824 4.7243 1.8061	5.0708 3.2864 36.616 0.9542 0.5491 4.2531	1.6223 0.3781 2.7542 1.2495 2.1082 28.139
9	60.838 4.7258 1.8063	5.0714 3.2865 36.617 0.9543 0.5491 4.2533	1.6224 0.3781 2.7544 1.2495 2.1082 28.139
	$P_1 = P_1^{(9)}$	$P_2 = P_2^{(9)}$	$P_3 = P_3^{(9)}$

III. NUMERICAL EXAMPLES

A real world physical example (a chemical plant model [14]) demonstrates the efficiency of the proposed method

$$A = 10^{-2}\begin{pmatrix} 95.407 & 1.9643 & 0.3597 & 0.0673 & 0.0190 \\ 40.849 & 41.317 & 16.084 & 4.4679 & 1.1971 \\ 12.217 & 26.326 & 36.149 & 15.930 & 12.383 \\ 4.1118 & 12.858 & 27.209 & 21.442 & 40.976 \\ 0.1305 & 0.5808 & 1.8750 & 3.6162 & 94.280 \end{pmatrix}$$

$$B^T = 10^{-2}\begin{pmatrix} 0.0434 & 2.6606 & 3.7530 & 3.6076 & 0.4617 \\ -0.0122 & -1.0453 & -5.5100 & -6.6000 & -0.9148 \end{pmatrix}$$

$$Q = I_5, \quad R = I_2.$$

These matrices are obtained from [14] by performing a discretization with a sampling rate $\Delta T = 0.5$. The small weakly coupling parameter ϵ is built into the problem and can be roughly estimated from the strongest coupled matrix (matrix B). The strongest coupling is in the third row, where

$$\epsilon = \frac{b_{31}}{b_{32}} = \frac{3.7530}{5.5100} \approx 0.68.$$

Simulation results are obtained using the L-A-S package for computer-aided control system [16] and presented in Table I.

For this specific real world example the proposed algorithm perfectly matches the presented theory since convergence, with the accuracy of 10^{-4}, is achieved after 9 iterations (i.e., $0.68^{18} = 10^{-4}$). Numerical examples performed in [7] for different values of ϵ support the proposed algorithm.

IV. CONCLUSION

A reduced-order optimal solution of the algebraic discrete weakly coupled Riccati equation is obtained. This result reduces off-line computational requirements and plays an important role in the design procedure of optimal and near-optimal controllers and filters for weakly coupled discrete systems.

APPENDIX I

It can be shown that

$$(I + A)^{-1} = \begin{pmatrix} O(1) & O(\epsilon) \\ O(\epsilon) & O(1) \end{pmatrix}. \tag{A.1}$$

Since S_d from (4b) and Q from (1) have the same weakly coupled structure as (A.1) so does D in (4d). The inverse of D is also of the weakly coupled form as defined in (A.1). From (4a) and (4c) the weakly coupled structure of matrices A_c and Q_c follows directly since they are given in terms of sums and/or products of weakly coupled matrices.

APPENDIX II

$$\Delta_1 = A_{11} - S_{11}P_1, \qquad \Delta_2 = A_{22} - S_{22}P_3$$

$$\Delta_{12} = A_{12} - S_{11}P_2 - S_{12}P_3, \qquad \Delta_{21} = A_{21} - S_{22}P_2^T - S_{12}^T P_1$$

$$M_1^{(j)} = P_2^{(j)}S_{12}^T P_1^{(j)} + P_1^{(j)}S_{12}P_2^{(j)^T} + P_2^{(j)}S_{22}P_2^{(j)^T} - P_2^{(j)}A_{21} - A_{21}^T P_2^{(j)^T} - \epsilon^2 E_1^{(j)}S_{11}E_1^{(j)}$$

$$M_3^{(j)} = P_3^{(j)}S_{12}^T P_2^{(j)} + P_2^{(j)^T}S_{12}P_3^{(j)} + P_2^{(j)^T}S_{11}P_2^{(j)} + P_3^{(j)}S_{12}^T P_3^{(j)} - P_2^{(j)^T}A_{12} - A_{12}^T P_2^{(j)} + \epsilon^2 E_3^{(j)}S_{22}E_3^{(j)}$$

$$M_2^{(j,j+1)} = P_2^{(j)}S_{12}^T P_2^{(j)} + \epsilon^2 E_1^{(j+1)}S_{11}E_2^{(j)} + \epsilon^2 E_2^{(j)}S_{22}E_3^{(j+1)} + \epsilon^2 E_1^{(j+1)}S_{12}E_3^{(j+1)}.$$

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