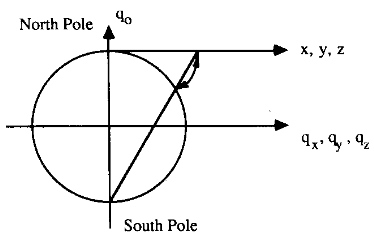

Fig. 2. Riemann's stereographic projection.

easily seen to be the Riemann metric

$$ds^2 = \frac{dx^2 + dy^2 + dz^2}{\left(1 + \frac{1}{4}(x^2 + y^2 + z^2)\right)^2}.$$

By analogy with the S^1 and S^2 cases, ds is called *arc length* element on the punctured sphere S^3. The distance $d(q_1, q_2)$ between two points q_1, q_2 on $S^3 - \{\text{south pole}\}$ is given by $\int_{q_1}^{q_2} ds$, the integral being evaluated along the geodesic line joining q_1 and q_2. By definition, the geodesic line minimizes $\int_{q_1}^{q_2} ds$. Computing the distance along the geodesic line indeed guarantees that $d(\cdot, \cdot)$ satisfies the triangle inequality, for

$$\min \int_{q_1}^{q_2} ds \le \min \int_{q_1}^{q_3} ds + \min \int_{q_3}^{q_2} ds.$$

Finally, we evaluate the distance between the north pole 1 and the point $q = (q_o, q_x, q_y, q_z)$. By the rotational invariance of the problem, we can assume that $q = (q_o, q_x, 0, 0)$. Since ds is the metric induced on $S^3 - \{\text{south pole}\}$ by the Euclidean metric on $\mathbb{R}^4$, the geodesic joining 1 and q is contained entirely in the (q_o, q_x) plane. Therefore

$$d(1, q) = \int_0^x \frac{dx}{1 + \frac{1}{4}x^2}$$

$$= 2\tan^{-1}\frac{x}{2}.$$

Finally, using (A.1) or doing some elementary geometry on Fig. 2, it is easily seen that $2\tan^{-1} x/2 = \cos^{-1} q_o$. Therefore

$$d(1, q) = \cos^{-1} q_o$$

as claimed.

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A New Version of the Chang Transformation

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***Abstract*—A new decoupling transformation is proposed for singularly perturbed linear systems. This transformation has the advantage over the previous one in that, in order to find such transformation, only one algorithm is required and the computation can be done in parallel since the transformation matrices are obtained from two independent equations having identical form.**

I. Introduction

A singularly perturbed system is represented by [1], [2]

$$\dot{u} = B_1 u + B_2 v, \tag{1}$$

$$\epsilon \dot{v} = B_3 u + B_4 v \tag{2}$$

where $u \in R^{m_1}$, $v \in R^{m_2}$ are slow and fast state variables, respectively, and matrices B_1, B_2, B_3, B_4 are of appropriate dimensions, which are constant in the case of time-invariant systems, and functions of time in the case of time-varying systems. A small parameter ϵ is positive.

The common approach to solve these systems is to first transform them into new coordinates such that the states are independent (decoupled) from each other. This leads to a block diagonal form which is easier to solve [3].

For the singularly perturbed system the corresponding transformation is [3]

$$T = \begin{bmatrix} I & -\epsilon P \\ -Q & I + \epsilon QP \end{bmatrix} \tag{3}$$

where P and Q are the solutions of the following two equations:

$$\epsilon \dot{P} = B_4 P - \epsilon P B_1 + \epsilon P B_2 P - B_3, \tag{4}$$

$$\epsilon \dot{Q} = \epsilon(B_1 - B_2 P)Q - Q(B_4 + \epsilon P B_2) - B_2. \tag{5}$$

Note that for the time-invariant case, the derivatives $\dot{P}$ and $\dot{Q}$ are zero.

The difficulty in solving (4) and (5) is that (5) can only be solved after the results of (4) are available. Therefore, computation must be done sequentially. Furthermore, two different algorithms are needed: one for (4) and the other for (5). This difficulty is overcome by introducing another transformation which decouples the original system as well as the transformation equations. This will enable us to compute P and Q in parallel and by using only one algorithm. The proposed transformations are extremely efficient, from the numerical point of view, in the case of time-varying systems since corresponding differential equations are completely decoupled. This is extremely important for singularly perturbed systems where both transformation equations (4) and (5) are stiff, and thus numerically ill-defined [2]. The main result of this note is given in the next section.

II. New Decoupling Transformation

Introducing the transformation

$$\alpha = u - \epsilon P v,$$
$$\beta = -Qu + v \tag{6}$$

Manuscript received June 8, 1990: revised September 21, 1990.
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IEEE Log Number 9107241.

0018-9286/92$03.00

and differentiating, we get

$$\dot{\alpha} = \dot{u} - \epsilon P\dot{v} - \epsilon \dot{P} v,$$
$$\dot{\beta} = -Q\dot{u} - \dot{Q}u + \dot{v}.$$

Substituting for $\dot{u}$ and $\dot{v}$ from the original system, and simplifying, we get

$$\dot{\alpha} = B_{10}\alpha - G_1(P)v \tag{7}$$

where

$$B_{10} = B_1 - PB_3$$

and

$$G_1(P) = \epsilon\dot{P} + PB_4 - \epsilon B_1 P - B_2 + \epsilon PB_3 P. \tag{8}$$

Also

$$\epsilon\dot{\beta} = B_{40}\beta - G_2(Q)u \tag{9}$$

where

$$B_{40} = B_4 - \epsilon QB_2$$

and

$$G_2(Q) = \epsilon\dot{Q} + \epsilon QB_1 - B_4 Q - B_3 + \epsilon QB_2 Q. \tag{10}$$

By setting $G_1(P) = 0$, and $G_2(Q) = 0$, we get the decoupled system

$$\dot{\alpha} = B_{10}\alpha = (B_1 - PB_3)\alpha, \tag{11}$$
$$\epsilon\dot{\beta} = B_{40}\beta = (B_4 - \epsilon QB_2)\beta \tag{12}$$

where P and Q can be calculated from the following two stiff differential equations:

$$\epsilon\dot{P} = -PB_4 + B_2 + \epsilon(B_1 P - PB_3 P), \tag{13}$$
$$\epsilon\dot{Q} = B_4 Q + B_3 + \epsilon(QB_1 + QB_2 Q). \tag{14}$$

The initial conditions for differential equations (13) and (14) are arbitrary [3], [4]. For time-invariant systems, equations (13) and (14) become algebraic ones. Efficient numerical methods for solving corresponding algebraic equations are discussed in [5]. Note that both (4) and (5) and (13) and (14) are stiff differential equations. They can be solved by using methods from [6]. It is known that due to a huge initial slope, solution of these equations requires a lot of time [6].

Thus, the introduced decoupling transformation is

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} I & -\epsilon P \\ -Q & I \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = T \begin{bmatrix} u \\ v \end{bmatrix}$$

where

$$T^{-1} = \begin{bmatrix} I + \epsilon PNQ & \epsilon PN \\ NQ & N \end{bmatrix}$$

with $N = (I - \epsilon QP)^{-1}$, and P and Q are the solutions of (13) and (14), respectively.

It is important to notice that in (4) and (5) one has to solve one Riccati and one Lyapunov equation sequentially. The total processing time in that case is greater than t_R, where t_R is the time for solving the Riccati equation. However, in (13) and (14) solutions of two Riccati equations are required, but due to parallelism the total processing time is t_R.

III. Conclusion

A different viewpoint is taken in developing the decoupling transformations for singularly perturbed linear systems. The proposed transformations have the advantage over the previous ones since they also decouple the transformation equations (13) and (14), enabling us to perform the computations in parallel. This is numerically very efficient for the case of time-varying systems where the corresponding differential equations are stiff.

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Geometric Theory for the Singular Roesser Model

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Abstract—**(A, E, B)-invariant and (E, A, B)-invariant subspaces for the two-dimensional singular Roesser model are investigated. These subspaces are related to the existence of the solutions when the boundary conditions are in these subspaces. Also existence of a solution sequence in certain subspaces derived from the invariant subspaces is shown. The boundary conditions that appear in the solution when some semistates in the solution are restricted to zero are also investigated.**

I. Introduction

Two-dimensional (2-D) state-space models have been studied extensively during the past decade and a half. During this time many 1-D state-space techniques [1], [2] have been generalized to their 2-D counterparts [3]-[6]. However, only a few publications have emerged considering the 2-D singular models, which are more general [7]-[10]. In fact, 2-D singular models deserve better consideration due to the physical motivations and their richer structure.

2-D system models may assume spatial parameters as well as time, consequently, they do not have any natural notion of causality. The notion of recursibility is a commonly assumed property for 2-D state-space models, and allows their solution. The 2-D singular models, however, do not require recursibility. This allows them to model systems whose states at any value of the parameters depend on data from any direction in the 2-D plane. For instance, the heat conduction problem over a finite plane, and a nonrecursible mask can be modeled as a singular, but not state-space 2-D systems [7], [10]. Also, the 2-D singular models allow algebraic constraints in addition to their dynamics, which is an improvement over state-space models.

In this note we consider geometric notions for the 2-D Roesser model (SRM). The geometric approach classifies system constraints and dynamics with respect to subspaces. In singular systems, where

Manuscript received June 15, 1990; revised February 15, 1991. This work was supported by the National Science Foundation under Grant ECS-8805932.

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IEEE Log Number 9107242.

0018-9286/92$03.00