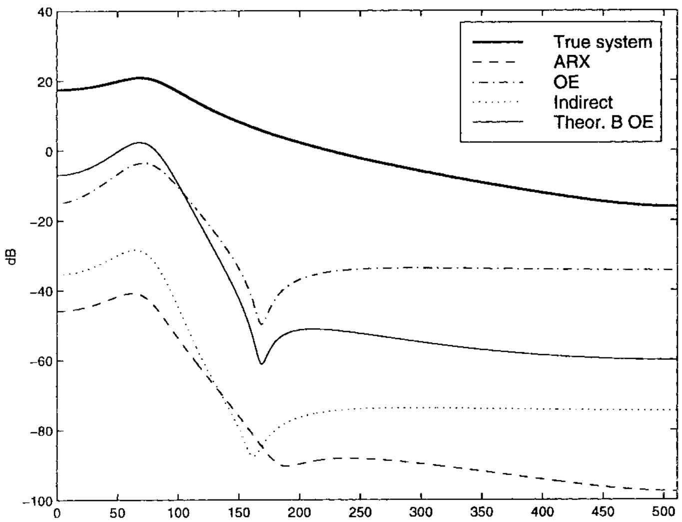

Fig. 1. Bode plot of the true system and mean errors.

The reference signal r and the noise e_0 were chosen as independent, zero mean, Gaussian white noise signals, with variances 1 $(=\Phi_r(\omega))$ and 0.01 $(=\lambda_0)$, respectively.

A Monte Carlo simulation consisting of 1024 different runs was performed. In each run we generated $N = 1024$ data points and identified the system directly using second-order ARX and output error models and indirectly using a second-order model of the kind (35) where we fixed $\bar{H}(q, \eta) = 1$. For the direct method with an output error method we can expect biased results since for this model $H(q, \theta) = 1 \neq H_0(q)$ [cf. (23)]. For the ARX model, on the other hand, there should ideally be no bias error since the chosen model structure coincides with the one used to generate the data. In Fig. 1 we plotted the true system together with the mean errors in the estimated models. The results for the ARX and the indirect cases are similar. The differences are likely to be due to numerical problems (poor initial conditions, problems with local minima, etc.) in the estimation routine used for the indirect method. These problems are not present in the ARX case since there the prediction error estimate is found without iterations by solving a standard least squares problem. In Fig. 1 we also included a plot of the theoretical bias error according to (22) for the output error model. As can be seen from the figure, the obtained bias error is close to the theoretical value.

VII. Conclusion

By studying the bias error due to feedback in the estimated transfer functions when using the direct method we have obtained a nonstandard motivation for the indirect approach to closed-loop identification. This method gives consistency regardless of the noise color but requires perfect knowledge of the regulator and gives suboptimal accuracy.

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Singular Perturbation Analysis of Cheap Control Problem for Sampled Data Systems

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***Abstract*—This paper studies the discrete-time cheap control problem for sampled data systems using the theory of singular perturbations. It is shown, by using the two time-scale property of singularly perturbed systems, that the problem can be solved in terms of two reduced-order subproblems for which computations can be done in parallel, thus increasing the computational speed. Similarly to the continuous-time case, the singular perturbation approach enables the decomposition of the algebraic Riccati equation into two reduced-order pure-slow and pure-fast continuous-time algebraic equations.**

***Index Terms*— Cheap control, decoupling, order reduction, sampled data systems, singular perturbations.**

I. Introduction

Cheap control refers to an optimal control problem in which the performance index includes only a small control cost. Its continuous-time version has been studied by a number of researchers (see, for example, [3], [6], [9], and references therein). However, the discrete-time cheap control problem, which occurs naturally when dealing

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with sampled data systems, has not been completely solved in the spirit of decomposing the problem into the slow and fast time scales and removing the problem's ill conditioning. A first approach in that direction can be found in [9], where a near optimal solution is presented. In this paper, we obtain the exact solution to the optimal cheap control problem for sampled data systems.

Consider the sampled data system

$$x(k+1) = (I + \varepsilon A)x(k) + \varepsilon Bu(k) \tag{1}$$

which can be obtained by uniformly sampling a continuous-time system with a sampling period ε, where ε is an arbitrary small positive number, $x \in \boldsymbol{R}^n$ are state variables, and $u \in \boldsymbol{R}^{n_2}$ is the control input. In addition, we assume that

$$B_a = \varepsilon B = \varepsilon \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \tag{2}$$

with $B \in \boldsymbol{R}^{n \times n_2}$, $B_2 \in \boldsymbol{R}^{n_2 \times n_2}$, and $\det(B_2) \neq 0$, which is required for the cheap control problem formulation [3], [9]. Under the above assumption, the system can be partitioned as

$$\begin{aligned} x_1(k+1) &= (I_{n_1} + \varepsilon A_1)x_1(k) + \varepsilon A_2 x_2(k) \\ x_2(k+1) &= \varepsilon A_3 x_1(k) + (I_{n_2} + \varepsilon A_4)x_2(k) + \varepsilon B_2 u(k). \end{aligned} \tag{3}$$

The cheap control problem for (1) is to determine the optimal control sequence $u(k)$ that minimizes the performance index

$$\begin{aligned} J &= \tfrac{1}{2} \sum_{n=0}^{\infty} [x^T(k)Qx(k) + \varepsilon^2 u^T(k)Ru(k)] \\ Q &= \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix} \geq 0, \qquad R > 0 \end{aligned} \tag{4}$$

with $Q_3 > 0$, which is the standard assumption for cheap control problems [3].

Note that the choice of (4) does not constrain the choice of the sampling period ε. We first perform sampling with any given ε, and then use that ε to scale the control penalty matrix in the performance criterion, which leads to the cheap control problem formulation. Note that R is an arbitrary positive definite matrix of $O(1)$, and the fact that the overall penalty matrix $\varepsilon^2 R$ is small indicates only that the control input is not expensive. We can use another small parameter ε_1 to indicate this fact as follows: take the control penalty matrix as $\varepsilon_1^2 R_1$, with $\varepsilon_1 = \alpha\varepsilon$ and $R_1 = R_1^T > 0$ an arbitrary matrix of $O(1)$. This leads to the control penalty matrix $\varepsilon^2 R$ with $R = \alpha^2 R_1$. To avoid the problem of dealing with two small parameters, we have adopted the form defined in (4), which has also been used in [9].[1]

[1] It should be emphasized that the presentation of this paper is valid within the framework of $O(\varepsilon)$ theory. In general, $O(\varepsilon^r)$ is defined by $O(\varepsilon^r) < \mathcal{K}\varepsilon^r$, where $\mathcal{K}$ is a bounded constant and r is any real number.

II. Decomposition of the Closed-Loop Cheap Control Problem

The decomposition of the cheap control problem is obtained by starting with the open-loop optimal solution of the optimization problem defined in (1)–(4), which is given by

$$u(k) = -\frac{1}{\varepsilon} R^{-1} B^T \lambda(k+1) \tag{5}$$

where λ is a costate variable satisfying [4]

$$\begin{bmatrix} x(k+1) \\ \lambda(k+1) \end{bmatrix} = \boldsymbol{H} \begin{bmatrix} x(k) \\ \lambda(k) \end{bmatrix}. \tag{6}$$

The matrix $\boldsymbol{H}$ is the standard Hamiltonian matrix, and has the following form [4]:

$$\boldsymbol{H} = \begin{bmatrix} A_a + \frac{1}{\varepsilon^2} B_a R^{-1} B_a^T A_a^{-T} Q & -\frac{1}{\varepsilon^2} B_a R^{-1} B_a^T A_a^{-T} \\ -A_a^{-T} Q & A_a^{-T} \end{bmatrix} \tag{7}$$

where

$$\begin{aligned} A_a &= \begin{bmatrix} I_{n_1} + \varepsilon A_1 & \varepsilon A_2 \\ \varepsilon A_3 & I_{n_2} + \varepsilon A_4 \end{bmatrix} \\ B_a R^{-1} B_a^T &= \varepsilon^2 \begin{bmatrix} 0 & 0 \\ 0 & B_2 R^{-1} B_2^T \end{bmatrix} \\ A_a^{-T} &= \begin{bmatrix} I_{n_1} + \varepsilon \overline{A_1} & \varepsilon \overline{A_2} \\ \varepsilon \overline{A_3} & I_{n_2} + \varepsilon \overline{A_4} \end{bmatrix} \\ A_a^{-T} Q &= \begin{bmatrix} \overline{Q_1} & \overline{Q_2} \\ \overline{Q_3} & \overline{Q_4} \end{bmatrix}. \end{aligned} \tag{8}$$

There is no need for analytical expressions of "bared" matrices; they have to be constructed by the computer in the process of calculations.

According to the above-defined partitions, the state–costate equations have the following expression:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \lambda_1(k+1) \\ \lambda_2(k+1) \end{bmatrix} = \boldsymbol{H} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \lambda_1(k) \\ \lambda_2(k) \end{bmatrix} \tag{9}$$

with the Hamiltonian matrix partitioned as shown in (10), at the bottom of the page.

Our goal is to put the state–costate system (9) and (10) into the singular perturbation form, and to achieve the pure slow–fast decomposition of the cheap control problem. We introduce the permutation matrix

$$E_1 = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & \varepsilon I_{n_1} & 0 \\ 0 & I_{n_2} & 0 & 0 \\ 0 & 0 & 0 & I_{n_2} \end{bmatrix} \tag{11}$$

$$\boldsymbol{H} = \begin{bmatrix} I_{n_1} + \varepsilon A_1 & \varepsilon A_2 & 0 & 0 \\ \varepsilon A_3 + B_2 R^{-1} B_2^T \overline{Q_3} & I_{n_2} + \varepsilon A_4 + B_2 R^{-1} B_2^T \overline{Q_4} & -\varepsilon B_2 R^{-1} B_2^T \overline{A_3} & B_2 R^{-1} B_2^T (I_{n_2} + \varepsilon \overline{A_4}) \\ -\overline{Q_1} & -\overline{Q_2} & I_{n_1} + \varepsilon \overline{A_1} & \varepsilon \overline{A_2} \\ -\overline{Q_3} & -\overline{Q_4} & \varepsilon \overline{A_3} & I_{n_2} + \varepsilon \overline{A_4} \end{bmatrix} \tag{10}$$

$$E_1 \boldsymbol{H} E_1^{-1} = \begin{bmatrix} I_{n_1} + \varepsilon A_1 & 0 & \varepsilon A_2 & 0 \\ -\varepsilon \overline{Q_1} & I_{n_1} + \varepsilon \overline{A_1} & -\varepsilon \overline{Q_2} & \varepsilon^2 \overline{A_2} \\ \varepsilon A_3 + B_2 R^{-1} B_2^T \overline{Q_3} & -B_2 R^{-1} B_2^T \overline{A_3} & I_{n_2} + \varepsilon A_4 + B_2 R^{-1} B_2^T \overline{Q_4} & B_2 R^{-1} B_2^T \times (I_{n_2} + \varepsilon \overline{A_4}) \\ -\overline{Q_3} & \overline{A_3} & -\overline{Q_4} & I_{n_2} + \varepsilon \overline{A_4} \end{bmatrix} \tag{12}$$

and set

$$\begin{bmatrix} p_1(k) \\ p_2(k) \end{bmatrix} = \begin{bmatrix} \varepsilon\lambda_1(k) \\ \lambda_2(k) \end{bmatrix}$$

to obtain the coordinate transformation

$$E_1 \begin{bmatrix} x_1(k) \\ x_2(k) \\ \lambda_1(k) \\ \lambda_2(k) \end{bmatrix} = \begin{bmatrix} x_1(k) \\ p_1(k) \\ x_2(k) \\ p_2(k) \end{bmatrix}.$$

In the transformed coordinates, the Hamiltonian matrix has the form shown in (12), at the bottom of the previous page. It is easy to observe that the transformed Hamiltonian matrix can be partitioned as

$$E_1 \boldsymbol{H} E_1^{-1} = \begin{bmatrix} I_{2n_1} + \varepsilon T_1 & \varepsilon T_2 \\ T_3 & T_4 \end{bmatrix} \tag{13}$$

with matrices T_1, T_2, T_3, and T_4 being given by

$$T_1 = \begin{bmatrix} A_1 & 0 \\ -\overline{Q_1} & \overline{A_1} \end{bmatrix}$$

$$T_2 = \begin{bmatrix} A_2 & 0 \\ -\overline{Q_2} & \varepsilon\overline{A_2} \end{bmatrix}$$

$$T_3 = \begin{bmatrix} \varepsilon A_3 + B_2R^{-1}B_2^T\overline{Q_3} & -B_2R^{-1}B_2^T\overline{A_3} \\ -\overline{Q_3} & \overline{A_3} \end{bmatrix}$$

$$T_4 = \begin{bmatrix} I_{n_2} + \varepsilon A_4 + B_2R^{-1}B_2^T\overline{Q_4} & -B_2R^{-1}B_2^T(I_{n_2} + \varepsilon\overline{A_4}) \\ -\overline{Q_4} & I_{n_2} + \varepsilon\overline{A_4} \end{bmatrix}.$$

Since the matrices T_1, T_2, T_3, T_4 are all of $O(1)$, the state–costate equations have, in the new coordinates, the standard form of a discrete singularly perturbed system [7], [8]:

$$\begin{aligned} \begin{bmatrix} x_1(k+1) \\ p_1(k+1) \end{bmatrix} &= (I_{2n_1} + \varepsilon T_1)\begin{bmatrix} x_1(k) \\ p_1(k) \end{bmatrix} + \varepsilon T_2 \begin{bmatrix} x_2(k) \\ p_2(k) \end{bmatrix} \\ \begin{bmatrix} x_2(k+1) \\ p_2(k+1) \end{bmatrix} &= T_3 \begin{bmatrix} x_1(k) \\ p_1(k) \end{bmatrix} + T_4 \begin{bmatrix} x_2(k) \\ p_2(k) \end{bmatrix}. \end{aligned} \tag{14}$$

It is important to observe that the matrix T_4 is the Hamiltonian matrix of the fast subsystem, and has no eigenvalues on the unit circle in the case when the stabilizability–detectability assumption is satisfied. Note that, from (8), we have $\overline{Q_4} = Q_3 + O(\varepsilon)$. Since

$$T_4 = T_4^{(0)} + O(\varepsilon) = \begin{bmatrix} I_{2n_2} & -B_2R^{-1}B_2^T \\ -Q_3 & I_{2n_2} \end{bmatrix} + O(\varepsilon)$$

the required assumption is as follows.

Assumption 1: The triple $(I, B_2, \sqrt{Q_3})$ is stabilizable–detectable.

Since we have already assumed that $\det(B_2) \neq 0$, and $Q_3 > 0$, the above assumption is satisfied; in other words, Assumption 1 is implied by the following assumption.

Assumption 2: $\det(B_2) \neq 0$ and $Q_3 > 0$.

To decouple slow and fast variables, we can now apply Chang's transformation [5], defined by

$$\mathcal{T} \begin{bmatrix} x_1(k) \\ p_1(k) \\ x_2(k) \\ p_2(k) \end{bmatrix} = \begin{bmatrix} \eta_1(k) \\ \xi_1(k) \\ \eta_2(k) \\ \xi_2(k) \end{bmatrix}$$

$$\begin{aligned} \mathcal{T} &= \begin{bmatrix} I_{2n_1} - \varepsilon HL & -\varepsilon H \\ L & I_{2n_2} \end{bmatrix} \\ \mathcal{T}^{-1} &= \begin{bmatrix} I_{2n_1} & \varepsilon H \\ -L & I_{2n_2} - \varepsilon LH \end{bmatrix} \end{aligned} \tag{15}$$

where matrices L and H satisfy the equations

$$\begin{aligned} H + T_2 - HT_4 + \varepsilon(T_1 - T_2L)H + \varepsilon HLT_2 &= 0 \\ -L + T_4L - T_3 - \varepsilon L(T_1 - T_2L) &= 0. \end{aligned} \tag{16}$$

The unique solution of (16) exists under the assumption that $(T_4 - I)$ is nonsingular, which is, as we have noted previously, satisfied for sufficiently small values of ε since the matrix $T_4^{(0)}$ has no eigenvalues on the unit circle. An algorithm for solving (16) is presented in [1].

Application of the transformation (15) results in a block-diagonal system matrix, that is,

$$\begin{aligned} &\mathcal{T}\begin{bmatrix} I_{2n_1} + \varepsilon T_1 & \varepsilon T_2 \\ T_3 & T_4 \end{bmatrix}\mathcal{T}^{-1} \\ &\quad = \begin{bmatrix} I_{2n_1} + \varepsilon T_1 - \varepsilon T_2L & 0 \\ 0 & T_4 + \varepsilon LT_2 \end{bmatrix} \end{aligned} \tag{17}$$

which corresponds to a singularly perturbed system in which slow and fast variables are completely decoupled [1]. Hence, in the new coordinates, we have

$$\begin{aligned} \begin{bmatrix} \eta_1(k+1) \\ \xi_1(k+1) \end{bmatrix} &= (I_{2n_1} + \varepsilon(T_1 - T_2L))\begin{bmatrix} \eta_1(k) \\ \xi_1(k) \end{bmatrix} \\ \begin{bmatrix} \eta_2(k+1) \\ \xi_2(k+1) \end{bmatrix} &= (T_4 + \varepsilon LT_2)\begin{bmatrix} \eta_2(k) \\ \xi_2(k) \end{bmatrix}. \end{aligned}$$

In the original coordinates, the required optimal solution has a closed-loop form

$$\lambda(k) = Px(k) \tag{18}$$

with P being the solution of the discrete algebraic Riccati equation

$$P = Q + A_a^TPA_a - A_a^TPB_a\left(\varepsilon^2R + B_a^TPB_a\right)^{-1}B_a^TPA_a. \tag{19}$$

The same is true in the transformed coordinates [1], that is, $\xi_1(k) = P_1\eta_1(k)$ and $\xi_2(k) = P_2\eta_2(k)$ or

$$\begin{bmatrix} \xi_1(k) \\ \xi_2(k) \end{bmatrix} = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}\begin{bmatrix} \eta_1(k) \\ \eta_2(k) \end{bmatrix}. \tag{20}$$

To use this fact, we rearrange the variables in (15) using the permutation matrix

$$E_2 = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & I_{n_2} & 0 \\ 0 & I_{n_1} & 0 & 0 \\ 0 & 0 & 0 & I_{n_2} \end{bmatrix} \tag{21}$$

so that the overall transformation between the original and new coordinates is $\Pi = E_2\mathcal{T}E_1$, which is partitioned as

$$\begin{bmatrix} \eta_1(k) \\ \eta_2(k) \\ \xi_1(k) \\ \xi_2(k) \end{bmatrix} = \begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_3 & \Pi_4 \end{bmatrix}\begin{bmatrix} x_1(k) \\ x_2(k) \\ \lambda_1(k) \\ \lambda_2(k) \end{bmatrix}. \tag{22}$$

Expression (20), together with (18) and (22), produces

$$\begin{aligned} \begin{bmatrix} \eta_1(k) \\ \eta_2(k) \end{bmatrix} &= \Pi_1\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \Pi_2\begin{bmatrix} \lambda_1(k) \\ \lambda_2(k) \end{bmatrix} \\ &= (\Pi_1 + \Pi_2P)\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \\ \begin{bmatrix} \xi_1(k) \\ \xi_2(k) \end{bmatrix} &= \Pi_3\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \Pi_4\begin{bmatrix} \lambda_1(k) \\ \lambda_2(k) \end{bmatrix} \\ &= (\Pi_3 + \Pi_4P)\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}. \end{aligned} \tag{23}$$

It follows from (20) and (23) that P_1 and P_2 can be expressed in terms of the solution of the global (full-order) Riccati equation (19) as

$$\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} = (\Pi_3 + \Pi_4 P)(\Pi_1 + \Pi_2 P)^{-1}. \tag{24}$$

Following the same logic, we can find P in terms of P_1 and P_2 by introducing the inverse transformation

$$\begin{bmatrix} \Omega_1 & \Omega_2 \\ \Omega_3 & \Omega_4 \end{bmatrix} = E_1^{-1} \mathcal{T}^{-1} E_2^T \tag{25}$$

which yields

$$P = \left(\Omega_3 + \Omega_4 \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}\right)\left(\Omega_1 + \Omega_2 \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}\right)^{-1}. \tag{26}$$

It is shown in Appendix A that the inversions defined in (24) and (26) exist for sufficiently small values of parameter ε.

To determine P_1 and P_2, we rewrite the decoupled system (17) in terms of its components

$$\begin{aligned} \begin{bmatrix} \eta_1(k+1) \\ \xi_1(k+1) \end{bmatrix} &= (I_{2n_1} + \varepsilon(T_1 - T_2 L)) \begin{bmatrix} \eta_1(k) \\ \xi_1(k) \end{bmatrix} \\ &= \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix} \begin{bmatrix} \eta_1(k) \\ \xi_1(k) \end{bmatrix} \end{aligned} \tag{27}$$

$$\begin{aligned} \begin{bmatrix} \eta_2(k+1) \\ \xi_2(k+1) \end{bmatrix} &= (T_4 + \varepsilon L T_2) \begin{bmatrix} \eta_2(k) \\ \xi_2(k) \end{bmatrix} \\ &= \begin{bmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{bmatrix} \begin{bmatrix} \eta_2(k) \\ \xi_2(k) \end{bmatrix}. \end{aligned} \tag{28}$$

Using these two decoupled state–costate systems (27) and (28), the reduced-order nonsymmetric *continuous-time*, respectively, pure-slow and pure-fast, algebraic Riccati equations are derived:

$$P_1 \alpha_1 - \alpha_4 P_1 - \alpha_3 + P_1 \alpha_2 P_1 = 0 \tag{29}$$

$$P_2 \beta_1 - \beta_4 P_2 - \beta_3 + P_2 \beta_2 P_2 = 0 \tag{30}$$

where matrices α_i and β_i are shown in (31) and (32), at the bottom of the page, with

$$L = \begin{bmatrix} L_1 & L_2 \\ L_3 & L_4 \end{bmatrix}, \qquad H = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix}. \tag{33}$$

The Riccati equation for the slow subsystem becomes

$$\begin{aligned} &P_1(A_1 - A_2 L_1) - (\overline{A_1} + \overline{Q_2} L_2 - \varepsilon \overline{A_2} L_4) P_1 + \overline{Q_1} \\ &\quad - \overline{Q_2} L_1 + \varepsilon A_2 L_3 - P_1 A_2 P_1 = 0. \end{aligned} \tag{34}$$

Note that $P_1 I_{n1}$ and $-I_{n1} P_1$ cancel out, and that the remaining terms are divided by ε.

The Riccati equation for the fast subsystem becomes

$$\begin{aligned} &P_2\left[B_2 R^{-1} B_2^T \overline{Q_4} + \varepsilon(A_4 + L_1 A_2 - L_2 \overline{Q_2})\right] - \varepsilon(\overline{A_4} + \varepsilon L_4 \overline{A_2}) P_2 \\ &\quad + P_2\left[-B_2 R^{-1} B_2^T (I_{n_2} + \varepsilon \overline{A_4}) + \varepsilon^2 L_2 \overline{A_2}\right] P_2 \\ &\quad + (\overline{Q_4} - L_3 A_2 + L_4 \overline{Q_2}) = 0. \end{aligned} \tag{35}$$

It is shown in Appendix B that an $O(\varepsilon)$ perturbation in (34) and (35) leads to the symmetric Riccati equations of the form

$$\begin{aligned} &P_1^{(0)}\left(A_1 - A_2 Q_3^{-1} Q_2^T\right) + \left(A_1 - A_2 Q_3^{-1} Q_2^T\right)^T P_1^{(0)} \\ &\quad + \left(Q_1 - Q_2 Q_3^{-1} Q_2^T\right) - P_1^{(0)} A_2 Q_3^{-1} A_2^T P_1^{(0)} = 0 \end{aligned} \tag{36}$$

$$\begin{aligned} P_2^{(0)} &= I_{n2} P_2^{(0)} I_{n2} + Q_3 - P_2^{(0)} B_2 \left(R + B_2^T P_2^{(0)} B_2\right)^{-1} \\ &\quad \cdot B_2^T P_2^{(0)}. \end{aligned} \tag{37}$$

The unique positive semidefinite stabilizing solution of (37) exists under Assumption 1. The existence of such a solution of the slow algebraic Riccati equation (36) requires the following assumption.

Assumption 3: The triple

$$\left(A_1 - A_2 Q_3^{-1} Q_2^T,\ \sqrt{A_2 Q_3^{-1} A_2^T},\ \sqrt{Q_1 - Q_2 Q_3^{-1} Q_2^T}\right)$$

is stabilizable–detectable.

The solutions of (36)–(37) can be used as very good initial guesses for the Newton method for solving the nonsymmetric Riccati equations (34) and (35).

Having determined the solution of the global Riccati equation in terms of solutions of the reduced-order algebraic Riccati equations using formula (26), we can now write the optimal control sequence using (5) and (18):

$$\begin{aligned} u(k) &= -\left(\varepsilon^2 R + B_a^T P B_a\right)^{-1} B_a^T P A_a x(k) \\ &= -\frac{1}{\varepsilon}\left(R + B^T P B\right)^{-1} B P A_a x(k). \end{aligned} \tag{38}$$

The advantage of the singular perturbation approach is the fact that completely decoupled slow and fast subsystems can be used for parallel processing of information. Using (22), we can rewrite (38) as

$$\begin{aligned} u(k) &= -\frac{1}{\varepsilon}\left(R + B^T P B\right)^{-1} B P A_a (\Pi_1 + \Pi_2 P)^{-1} \begin{bmatrix} \eta_1(k) \\ \eta_2(k) \end{bmatrix} \\ &= -\frac{1}{\varepsilon} F_1 \eta_1(k) - \frac{1}{\varepsilon} F_2 \eta_2(k) \end{aligned} \tag{39}$$

with $\eta_1(k)$ and $\eta_2(k)$ obtained from the pure-slow and pure-fast subsystems, respectively, given by

$$\eta_1(k+1) = (\alpha_1 + \alpha_2 P_1) \eta_1(k) \tag{40}$$

$$\eta_2(k+1) = (\beta_1 + \beta_2 P_2) \eta_2(k). \tag{41}$$

The optimal gains $F_1 \in \boldsymbol{R}^{n_2 \times n_1}$ and $F_2 \in \boldsymbol{R}^{n_2 \times n_2}$ are obtained by appropriately partitioning the gain matrix defined in (39).

In summary, we have established the following theorem.

Theorem: Under the conditions stated in Assumptions 1–3, there exists a nonsingular transformation $\Pi_1 + \Pi_2 P$ such that

$$\begin{bmatrix} \eta_1(k) \\ \eta_2(k) \end{bmatrix} = (\Pi_1 + \Pi_2 P) \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

decouples the original system into pure-slow and pure-fast subsystems (40) and (41), where P_1 and P_2 are the unique solutions of the exact pure-slow and pure-fast completely decoupled algebraic Riccati equations (29) and (30). Even more, the global solution P can be obtained from the reduced-order exact pure-slow and pure-fast

$$\begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix} = \begin{bmatrix} I_{n1} + \varepsilon(A_1 - A_2 L_1) & -\varepsilon A_2 L_2 \\ \varepsilon(-\overline{Q_1} + \overline{Q_2} L_1 - \varepsilon A_2 L_3) & I_{n1} + \varepsilon(\overline{A_1} + \overline{Q_2} L_2 - \varepsilon \overline{A_2} L_4) \end{bmatrix} \tag{31}$$

$$\begin{bmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{bmatrix} = \begin{bmatrix} I_{n2} + \varepsilon A_4 + B_2 R^{-1} B_2^T \overline{Q_4} + \varepsilon(L_1 A_2 - L_3 \overline{Q_2}) & -B_2 R^{-1} B_2^T (I_{n2} + \varepsilon \overline{A_4}) + \varepsilon^2 L_2 \overline{A_2} \\ -\overline{Q_4} + L_3 A_2 - L_4 \overline{Q_2} & I_{n2} + \varepsilon \overline{A_4} + \varepsilon^2 L_4 \overline{A_2} \end{bmatrix} \tag{32}$$

algebraic regulator Riccati equations using formula (26). Known matrices Ω_i, $i = 1, 2, 3, 4$, and Π_1, Π_2 are given in terms of the solutions of the Chang decoupling equations (16).

As pointed out by an anonymous reviewer, the condition stated in Assumption 2, $\det(B_2) \neq 0$, can be replaced by the assumption that the pair (A, B) is controllable and the sampling period is not pathologic.

A numerical example that demonstrates the efficiency of the proposed method can be found in [10].

III. Conclusions

This paper presents the cheap control problem associated with sampled data systems. Using the theory of singular perturbations, the required optimal solution has been obtained in terms of pure-slow and pure-fast subsystems, which have been constructed through parallel computations. High dimensionality and ill conditioning of the original problem are eliminated, and the computational speed is increased. The discrete-time algebraic Riccati equation associated with the cheap control problem has been decomposed into two reduced-order nonsymmetric continuous-time algebraic Riccati equations which are easily solvable.

Appendix A

For the matrices involved in (24), we can write

$$\Pi = E_2 \mathcal{T} E_1 = \begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_3 & \Pi_4 \end{bmatrix} = \begin{bmatrix} I_{n1} & 0 & 0 & 0 \\ L_1 & I_{n_2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ L_3 & 0 & 0 & I_{n_2} \end{bmatrix} + O(\varepsilon)$$

from which we get

$$\Pi_1 = \begin{bmatrix} I_{n_1} & 0 \\ L_1 & I_{n_2} \end{bmatrix} + O(\varepsilon), \qquad \Pi_2 = \begin{bmatrix} O(\varepsilon^2) & O(\varepsilon) \\ O(\varepsilon) & 0 \end{bmatrix}.$$

Taking into account the partition for the solution of the Riccati equation which is given by [9]

$$P = \begin{bmatrix} \frac{1}{\varepsilon} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}$$

we get

$$\Pi_1 + \Pi_2 P = \begin{bmatrix} I_{n_1} & 0 \\ L_1 + L_2 P_{11} & I_{n_2} \end{bmatrix} + O(\varepsilon)$$

which proves that $\Pi_1 + \Pi_2 P$ is always invertible for small values of ε.

Similarly, for the matrices involved in (26), we can write

$$\Omega = E_1^{-1} \mathcal{T}^{-1} E_2^T = \begin{bmatrix} \Omega_1 & \Omega_2 \\ \Omega_3 & \Omega_4 \end{bmatrix} = \begin{bmatrix} I_{n1} & 0 & 0 & 0 \\ -L_1 & I_{n_2} & -L_2 & 0 \\ 0 & H_3 & \frac{1}{\varepsilon} I_{n_1} & H_4 \\ -L_3 & 0 & -L_4 & I_{n_2} \end{bmatrix} + O(\varepsilon)$$

from which we have

$$\Omega_1 = \begin{bmatrix} I_{n_1} & 0 \\ -L_1 & I_{n_2} \end{bmatrix} + O(\varepsilon), \qquad \Omega_2 = \begin{bmatrix} 0 & 0 \\ -L_2 & 0 \end{bmatrix} + O(\varepsilon).$$

It follows that

$$\Omega_1 + \Omega_2 \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} = \begin{bmatrix} I_{n_1} & 0 \\ -L_1 - L_2 P_1 & I_{n_2} \end{bmatrix} + O(\varepsilon)$$

which proves that

$$\Omega_1 + \Omega_2 \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$$

is also invertible for small values of ε.

Appendix B

An $O(\varepsilon)$ approximation for A_a^{-1} is

$$A_{a\varepsilon}^{-1} = \begin{bmatrix} I_{n_1} - \varepsilon A_1 & \varepsilon A_2 \\ \varepsilon A_3 & I_{n_2} - \varepsilon A_4 \end{bmatrix}$$

from which the $O(\varepsilon)$ approximation for A_a^{-T} is

$$A_{a\varepsilon}^{-T} = \begin{bmatrix} I_{n_1} + \varepsilon \overline{A_{1\varepsilon}} & \varepsilon \overline{A_{2\varepsilon}} \\ \varepsilon \overline{A_{3\varepsilon}} & I_{n_2} + \varepsilon \overline{A_{4\varepsilon}} \end{bmatrix} = \begin{bmatrix} I_{n_1} - \varepsilon A_1^T & \varepsilon A_2^T \\ \varepsilon A_3^T & I_{n_2} - \varepsilon A_4^T \end{bmatrix}.$$

This implies that $O(\varepsilon)$ perturbation in the matrices $\overline{A_1}$, $\overline{A_2}$, $\overline{A_3}$, $\overline{A_4}$ results in the following matrices:

$$\overline{A_{1\varepsilon}} = -A_1^T, \qquad \overline{A_{2\varepsilon}} = A_3^T, \qquad \overline{A_{3\varepsilon}} = A_2^T, \qquad \overline{A_{4\varepsilon}} = -A_4^T.$$

An $O(\varepsilon)$ approximation for $A_a^{-T}Q$ produces

$$\begin{aligned} A_a^{-T} Q &= \begin{bmatrix} I_{n_1} + \varepsilon \overline{A_1} & \varepsilon \overline{A_2} \\ \varepsilon \overline{A_3} & I_{n_2} + \varepsilon \overline{A_4} \end{bmatrix} \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix} \\ &= \begin{bmatrix} (I_{n_1} + \varepsilon \overline{A_1}) Q_1 + \varepsilon \overline{A_2} Q_2^T & (I_{n_1} + \varepsilon \overline{A_1}) Q_2 + \varepsilon \overline{A_2} Q_3 \\ \varepsilon \overline{A_3} Q_1 + (I_{n_2} + \varepsilon \overline{A_4}) Q_2^T & \varepsilon \overline{A_3} Q_2 + (I_{n_2} + \varepsilon \overline{A_4}) Q_3^T \end{bmatrix} \\ &= \begin{bmatrix} \overline{Q_1} & \overline{Q_2} \\ \overline{Q_3} & \overline{Q_4} \end{bmatrix} = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix} + O(\varepsilon). \end{aligned}$$

Hence, an $O(\varepsilon)$ perturbation leads to

$$\overline{Q_{1\varepsilon}} = Q_1, \qquad \overline{Q_{2\varepsilon}} = Q_2, \qquad \overline{Q_{3\varepsilon}} = Q_2^T, \qquad \overline{Q_{2\varepsilon}} = Q_3.$$

An $O(\varepsilon)$ perturbation in the Chang transformation results in

$$\mathcal{T}_\varepsilon = \begin{bmatrix} I_{2n_1} & 0 \\ L_\varepsilon & I_{2n_2} \end{bmatrix}, \qquad \mathcal{T}_\varepsilon^{-1} = \begin{bmatrix} I_{2n_1} & 0 \\ -L_\varepsilon & I_{2n_2} \end{bmatrix}$$

with L_ε satisfying

$$-L_\varepsilon + T_{4\varepsilon} L_\varepsilon - T_{3\varepsilon} = 0$$

from which

$$L_\varepsilon = (T_{4\varepsilon} - I)^{-1} T_{3\varepsilon}$$

with matrices $T_{3\varepsilon}$ and $T_{4\varepsilon}$ representing $O(\varepsilon)$ approximations for T_3 and T_4. We get

$$T_{3\varepsilon} = \begin{bmatrix} B_2 R^{-1} B_2^T Q_2^T & -B_2 R^{-1} B_2^T A_2^T \\ -Q_2^T & A_2^T \end{bmatrix}$$

$$T_{4\varepsilon} - I = \begin{bmatrix} B_2 R^{-1} B_2^T Q_3 & -B_2 R^{-1} B_2^T Q_2 \\ -Q_3 & 0 \end{bmatrix}$$

$$(T_{4\varepsilon} - I)^{-1} = \begin{bmatrix} 0 & -Q_3^{-1} \\ -B_2^{-T} R B_2^{-1} & -I \end{bmatrix}.$$

Thus, an $O(\varepsilon)$ approximation for L is

$$L_\varepsilon = \begin{bmatrix} Q_3^{-1} Q_2^T & Q_3^{-1} A_2^T \\ 0 & 0 \end{bmatrix}.$$

Now, we plug all of these $O(\varepsilon)$ approximations in the slow and fast nonsymmetric Riccati equations (34) and (35) and neglect all terms which are of $O(\varepsilon)$.

For the slow equation (34), we have

$$\begin{aligned} A_1 - A_2 L_\varepsilon &= A_1 - A_2 Q_3^{-1} Q_2^T \\ \overline{A_{1\varepsilon}} + \overline{Q_{2\varepsilon}} L_{2\varepsilon} &= -\left(A_1 - A_2 Q_3^{-1} Q_2^T\right) \\ \overline{Q_{1\varepsilon}} - \overline{Q_{2\varepsilon}} L_{1\varepsilon} &= Q_1 - Q_2 Q_3^{-1} Q_2^T \\ A_2 L_{2\varepsilon} &= A_2 Q_3^{-1} A_2^T \end{aligned}$$

which yields the symmetric continuous-time algebraic Riccati equation

$$P_1^{(0)}\left(A_1 - A_2Q_3^{-1}Q_2^T\right) + \left(A_1 - A_2Q_3^{-1}Q_2^T\right)^T P_1^{(0)} + \left(Q_1 - Q_2Q_3^{-1}Q_2^T\right) + P_1^{(0)}A_2Q_3^{-1}A_2^TP_1^{(0)} = 0.$$

It is interesting to observe that this equation is identical to the slow approximate algebraic Riccati equation of [9].

For the fast Riccati equation (35), we get an $O(\varepsilon)$ approximation as follows:

$$P_2^{(0)}B_2R^{-1}B_2^TQ_3 + Q_3 - P_2^{(0)}B_2R^{-1}B_2^TP_2^{(0)} = 0$$

which can be rewritten as

$$0 = -Q_3 + P_2^{(0)}B_2\left(R + B_2^TP_2^{(0)}B_2\right)^{-1}B_2^TP_2^{(0)}$$

which is the discrete-time algebraic Riccati equation corresponding to the fast variables of [9].

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$\mathcal{H}_2$ Guaranteed Cost-Switching Surface Design for Sliding Modes with Nonmatching Disturbances

Richardo H. C. Takahashi and Pedro L. D. Peres

***Abstract*—This note presents an extension of Utkin and Yang's method for sliding surface design via the minimization of a quadratic performance criterion. The method proposed here takes into account nonmatching disturbance signals and nonmatching system model uncertainties. An upper bound for the $\mathcal{H}_2$ norm of the disturbance-to-output transfer function is minimized, leading to a guaranteed cost $\mathcal{H}_2$ controller.**

***Index Terms*—$\mathcal{H}_2$ guaranteed cost, nonmatching disturbances, sliding modes, uncertain systems.**

Notation

$\mathcal{R}(\cdot)$	Range space (for matrices) or the image (for functions) of the argument, depending on the context.
$\mathcal{N}(\cdot)$	Null space of the argument.
$\rho(\cdot)$	Rank of the argument.
$(\cdot)'$	Transpose of the argument.
$(\cdot)^*$	Complex conjugate transpose of the argument.
$\mathcal{E}(\cdot)$	Mathematical expectation of the argument.
$\text{tr}(\cdot)$	Trace of the square matrix argument.

I. Problem Statement

This note deals with the design of sliding surfaces for sliding mode control systems. Systems with convex-bounded nonmatching model parameter uncertainties and nonmatching disturbance inputs are considered. The design problem is formulated in terms of *linear matrix inequalities* (LMI's) with the minimization of an upper bound for the system $\mathcal{H}_2$ norm in the uncertainty set. As a byproduct, the quadratic stability of the closed-loop system is guaranteed.

For brevity, there is no presentation of the background of these theories, and the reader is referred to [1]–[3] for further details on the sliding mode theory, and to [4] and [5] for more information on the convex optimization approach for $\mathcal{H}_2$ optimal control with convex-bounded uncertainties.

In the paper by Utkin and Yang [6] (and also in [7]), a nominally linear system is considered:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times r}$ are, respectively, the nominal dynamic matrix and the control input matrix. This system will be controlled with a sliding mode controller which constrains the state vector to a linear surface (sliding surface) of dimension $n - r$ after a finite transient time τ:

$$\begin{aligned} Cx(t) &= \mathbf{0}, \qquad \forall t \geq \tau \\ \rho(C) &= r. \end{aligned} \tag{2}$$

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