Near-Optimum Regulators for Stochastic Linear Singularly Perturbed Systems

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***Abstract*—This paper presents a new approach to the decomposition and approximation of linear-quadratic-Gaussian estimation and control problems for singularly perturbed systems. The Kalman filter is decomposed into separate slow-mode and fast-mode filters via the use of a decoupling transformation. A near-optimal control law is derived by approximating the coefficients of the optimal control law. The order of approximation of the optimal performance is $0(\mu^N)$ where N is the order of approximation of the coefficients.**

I. Introduction

THE singular perturbation approach [1] to linear quadratic (LQ) regulator problems of systems having slow and fast modes has led to useful and attractive approximation methods which are well-documented [2]–[8]. There are two different procedures for deriving approximations of the LQ optimal solution. In the first one, the solution of the regulator Riccati equation is obtained as an asymptotic, or power series, expansion in the perturbation parameter μ (cf., [3], [5]). Approximate feedback control laws are derived by truncating the asymptotic expansions of the feedback coefficients of the optimal control law. Such approximations have been shown to be near-optimal with performance that can be made as close to the optimal performance as desired by including enough terms in the truncated expansions. The second procedure for deriving approximate feedback control laws is based on formal slow–fast decompositions of the LQ problem [8]. Two lower order LQ problems are defined for the slow and fast variables leading to slow and fast feedback control laws. A composite feedback control law is formed as the sum of the slow and fast controls. It has been shown [8] that the feedback coefficients in the composite control law are the zero-order terms of the expansions of the optimal feedback coefficients. Thus, the second procedure is also near-optimal as $\mu \to 0$, although the best it can do is to achieve a performance which is $0(\mu^2)$ close to the optimal performance. Extending these approximation procedures to the linear-quadratic-Gaussian (LQG) problem has not been an easy task. Although the duality of the filter Riccati equation and the regulator Riccati equation can be used, together with the results of [3]–[5], to obtain asymptotic approximations to the filter gains, such approximations will not be satisfactory because they only reduce the off-line computational effort of computing the filter gains, but they do not help the on-line computations of implementing the Kalman filter which will be of the same order of the overall singularly perturbed system. Because of the slow–fast nature of the variables in singularly perturbed systems it has been felt that the Kalman filter need not be implemented as one whole filter; rather it may be replaced by two lower order filters which separately estimate the slow and fast variables and are implemented in different time scales. That conviction has motivated the two previous approaches for approximating LQG optimal control. The first approach [9], [10] extends the composite control idea of deterministic LQ problems to stochastic LQG problems. Formal lower order LQG problems are defined for the slow and fast variables leading to slow and fast feedback control laws which employ slow-mode and fast-mode filters, respectively. The slow and fast controls are added together to get a composite feedback control which has been shown [10] to be near-optimal by showing that the performance criterion, under the composite control law, converges to the optimal performance as $\mu \to 0$. The technical details of the approach have some difficulties arising mainly from trying to extend deterministic notions of singularly perturbed techniques [1], like boundary layers, to systems driven by white noise. Those difficulties affect the results; for example, in the filtering result of [9] the estimates of the fast variables are formally approximated by white noise, which, as pointed out in [9], makes sense only after integrating the fast variables over an interval of time of fixed length. The formal way of introducing the composite control law has two undesirable consequences. First, there is no clear link between the composite and optimal control laws. Recall that in the deterministic LQ regulator even though the composite control law is derived through formal slow–fast decompositions, it is known that its feedback coefficients are the zero-order terms of the expansions of the optimal feedback coefficients. There is no corresponding link in the LQG case. Notice that the composite control law comprises two lower order filters while the optimal control law comprises a full-order filter so that direct comparison of the filter coefficients is not feasible. Second because of the formal nature of introducing the composite control law there are no routines for improving the approximation. Such routines are available in the LQ problem [3], [5]. The second approach to approximate the optimal solution of the LQG problem [11] starts by writing the closed-loop equations of the optimal system as a system driven by white noise, i.e., after eliminating the input and output variables by substituting them in terms of the state variables of the plant and the Kalman filter. The entry of the filter's fast variables in its slow variable equations is eliminated by formally setting $\mu = 0$ on the left-hand side of the filter equations. Such elimination results in a slow-mode filter which is implemented at a slower rate leading to on-line reductions [11]. The approximation has been justified [11] by showing that the state of the plant under the approximate control law converges to the state of the plant under optimal control as $\mu \to 0$. The main difficulty with the approach is that the suggested approximation does not have the intuitively appealing slow–fast decomposition structure of the composite control law of [10]. Even the separation between the regulation and estimation tasks, which is preserved in the composite control law of [10], is lost since the approximation is defined by manipulating the closed-loop equations. Moreover, as in the first approach, there is no mechanism for improving approximations.

This paper presents a new approach to the decomposition and approximation of LQG optimal control for singularly perturbed systems. The new approach alleviates the difficulties of the previous approaches, is conceptually simple, and retains the physically motivated structure of the composite control of [10]. In the new approach the decomposition and approximation tasks are sep-

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arated from each other. Decomposing the Kalman filter into slow-mode and fast-mode filters is achieved via the use of a decoupling transformation that has been introduced in [12] for block diagonalization of singularly perturbed systems. Approximating the feedback control law is then achieved by truncating expansions of the coefficients in μ, where the coefficients are analytic in μ at $\mu=0$. The resulting feedback control law is shown to be a near-optimal solution of the LQG problem by studying the closed-loop system as a system driven by white noise. In that study, standard variance analysis is employed to reduce the problem to one of studying a deterministic singularly perturbed equation which can be easily handled using the well-known singular perturbation techniques [1].

The paper is organized as follows. In Section II we study the approximation of singularly perturbed systems driven by white noise. It is shown that an Nth-order approximation in which the system coefficients and initial conditions are $0(\mu^N)$ close to the exact ones is a well-defined and valid approximation in the sense that the differences between the exact and approximate solutions are $0(\mu^N)$ in the case of slow variables, and $0(\mu^{N-1/2})$ in the case of fast variables, where the order of approximation is taken in mean square.[1] In Section III, the LQG regulator is considered. A decoupling transformation is used to represent the Kalman filter in new coordinates in which the slow and fast variables are decoupled. An Nth-order approximate feedback control law is defined by truncating expansions of coefficients. A study of the resulting closed-loop system, employing the results of Section II, shows that the relative increase in the performance criterion over its optimal value is $0(\mu^N)$. A simplified first-order approximation is defined and shown to be equivalent to the near-optimal control of [10]. Section IV contains a numerical example and Section V includes discussions of various aspects of the LQG approximations and compares them to the corresponding LQ approximations.

II. Approximation of Singularly Perturbed Systems Driven by White Noise

Consider the linear time-invariant singularly perturbed system

$$\dot{x}(t)=A(\mu)x(t)+B(\mu)y(t)+E(\mu)w(t),\quad x(0)=x^0(\mu), \tag{2.1}$$

$$\mu\dot{y}(t)=C(\mu)x(t)+D(\mu)y(t)+F(\mu)w(t),\quad y(0)=y^0(\mu), \tag{2.2}$$

where $x\in R^n$, $y\in R^m$, $w\in R^r$, and μ is a small positive scalar parameter. The system matrices are analytic functions in μ at $\mu=0$, i.e.,

$$A(\mu)=\sum_{i=0}^{\infty}\frac{\mu^i}{i!}A_i \tag{2.3}$$

with similar expansions for B, C, D, E, and F. The input $w(t)$ is zero-mean, stationary, white Gaussian noise with intensity matrix $V>0$, i.e.,

$$E\{w(t)w^T(s)\}=V\delta(t-s). \tag{2.4}$$

The initial conditions $x^0(\mu)$ and $y^0(\mu)$ are Gaussian random vectors with means $\bar{x}^0(\mu)$ and $\bar{y}^0(\mu)$, and joint variance matrix $\Gamma^0(\mu)$, where $\bar{x}^0(\mu)$, $\bar{y}^0(\mu)$, and $\Gamma^0(\mu)$ are analytic in μ at $\mu=0$ with expansions similar to (2.3). It is assumed that the matrices $(A_0-B_0D_0^{-1}C_0)$ and D_0 are Hurwitz, i.e.,

$$\mathrm{Re}\,\lambda(D_0)<0 \tag{2.5}$$

and

$$\mathrm{Re}\,\lambda(A_0-B_0D_0^{-1}C_0)<0. \tag{2.6}$$

Conditions (2.5) and (2.6) guarantee [13] that for sufficiently small μ the singularly perturbed system is asymptotically stable. The purpose of this section is to study approximations of $x(t)$ and $y(t)$ when μ is small. We are interested in approximations $x_N(t)$, $y_N(t)$ which are defined by the following equations:

$$\dot{x}_N(t)=A^N(\mu)x_N(t)+B^N(\mu)y_N(t)+E^N(\mu)w(t),\quad x_N(0)=x_N^0(\mu), \tag{2.7}$$

$$\mu\dot{y}_N(t)=C^N(\mu)y_N(t)+D^N(\mu)y_N(t)+F^N(\mu)w(t),\quad y_N(0)=y_N^0(\mu). \tag{2.8}$$

The matrices $A^N(\mu)$ through $F^N(\mu)$ are analytic functions in μ which are $0(\mu^N)$ close to the corresponding matrices $A(\mu)$ through $F(\mu)$, respectively, e.g.,

$$A^N(\mu)=\sum_{i=0}^{\infty}\frac{\mu^i}{i!}A_i^N,\qquad A_i^N=\begin{cases}A_i & \forall i\in[0,N-1].\\ \text{arbitrary } i\geqslant N\end{cases} \tag{2.9}$$

The initial conditions $x_N^0(\mu)$ and $y_N^0(\mu)$ are Gaussian random vectors with means $\bar{x}_N^0(\mu)$ and $\bar{y}_N^0(\mu)$ and joint variance matrix $\Gamma_N^0(\mu)$, where $\bar{x}_N^0(\mu)$, $\bar{y}_N^0(\mu)$, and $\Gamma_N^0(\mu)$ are analytic in μ at $\mu=0$. It is assumed that

$$E\left\{\begin{pmatrix}x_N^0(\mu)-x^0(\mu)\\ y_N^0(\mu)-y^0(\mu)\end{pmatrix}\begin{pmatrix}x_N^0(\mu)-x^0(\mu)\\ y_N^0(\mu)-y^0(\mu)\end{pmatrix}^T\right\}=\sum_{i=2N}^{\infty}\frac{\mu^i}{i!}\Gamma_i=0(\mu^{2N}). \tag{2.10}$$

In other words, the approximation (2.7), (2.8) is obtained by making $0(\mu^N)$ perturbations in the matrix coefficients and initial conditions of (2.1), (2.2), where the perturbation in initial conditions is taken in mean square sense. In order to validate approximating $x(t)$ and $y(t)$ by $x_N(t)$ and $y_N(t)$, we study the variances[2] of $(x(t)-x_N(t))$ and $(y(t)-y_N(t))$ as $\mu\to 0$. We also study evaluation of quadratic forms in x and y like

$$\sigma=\lim_{t\to\infty}E\left\{\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}^T(H(\mu),J(\mu))^T(H(\mu),J(\mu))\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}\right\} \tag{2.11}$$

where $H(\mu)$ and $J(\mu)$ are analytic functions in μ. Such quadratic forms will appear in the steady-state LQG control problem (Section III). We examine approximating σ by σ_N which is given by

[1]A random variable $f(\mu)$ is said to be $0(\mu)$ in the mean square sense if there exist $\mu^*>0$ and $K>0$ such that $\forall\mu\in(0,\mu^*)$, $(Ef^2(\mu))^{1/2}\leqslant K\mu$.

[2]It is sufficient to study the variances of the errors since their means are known to be $0(\mu^N)$ by deterministic singular perturbation results [1].

$$\sigma_N = \lim_{t\to\infty} E\left\{\begin{pmatrix} x_N(t) \\ y_N(t)\end{pmatrix}^T (H^N(\mu), J^N(\mu))^T \cdot (H^N(\mu), J^N(\mu))\begin{pmatrix} x_N(t) \\ y_N(t)\end{pmatrix}\right\} \quad (2.12)$$

where $H^N(\mu)$ and $J^N(\mu)$ are analytic functions in μ which are $0(\mu^N)$ close to $H(\mu)$ and $J(\mu)$, respectively.

The results of this section are given in the following two theorems which are stated under the above conditions (for steady-state results (2.10) is not required).

Theorem 1: For all $t(0 \leqslant t < \infty)$

$$\operatorname{var}(x(t) - x_N(t)) = 0(\mu^{2N}), \quad (2.13)$$

$$\operatorname{var}(y(t) - y_N(t)) = 0(\mu^{2N-1}) \quad (2.14)$$

and

$$E\{[(x(t)-x_N(t)) - E(x(t)-x_N(t))] \cdot [(y(t)-y_N(t)) - E(y(t)-y_N(t))]^T\} = 0(\mu^{2N}). \quad (2.15)$$

Furthermore, if the initial condition closeness assumption (2.10) is not satisfied, (2.13)–(2.15) hold at steady state, i.e., as $t \to \infty$.

Theorem 1 establishes that $(x_N(t), y_N(t))$ is a valid approxiamtion to $(x(t), y(t))$. The mean square order of approximation of x is $0(\mu^N)$, which is the order of perturbation of parameters and initial conditions, but the mean square order of approximation of y is only $0(\mu^{N-1/2})$. It is emphasized that Theorem 1 holds even though the variances of $y(t)$ and $y_N(t)$ could be $0(1/\mu)$ because of the presence of white noise input multiplied by $1/\mu$.

Theorem 2:

$$\frac{\Delta\sigma}{\sigma} = 0(\mu^N) \quad (2.16)$$

where $\Delta\sigma = \sigma_N - \sigma$.

Again, we emphasize that Theorem 2 holds even though both σ and σ_N, could be $0(1/\mu)$.

Proof of Theorem 1: Let

$$e_x = x - x_N \quad \text{and} \quad e_y = y - y_N.$$

The variances of e_x and e_y can be determined by studying the following system of equations:

$$\begin{bmatrix} \dot x(t) \\ \dot e_x(t) \\ \dot y(t) \\ \dot e_y(t)\end{bmatrix} = \begin{bmatrix} A & 0 & B & 0 \\ \Delta A & A^N & \Delta B & B^N \\ \frac{1}{\mu}C & 0 & \frac{1}{\mu}D & 0 \\ \frac{1}{\mu}\Delta C & \frac{1}{\mu}C^N & \frac{1}{\mu}\Delta D & \frac{1}{\mu}D^N \end{bmatrix}\begin{bmatrix} x(t) \\ e_x(t) \\ y(t) \\ e_y(t)\end{bmatrix} + \begin{bmatrix} E \\ \Delta E \\ \frac{1}{\mu}F \\ \frac{1}{\mu}\Delta F\end{bmatrix} w(t), \quad (2.17)$$

where $\Delta A = A - A^N$, $\Delta B = B - B^N$, etc.

For shorthand notation, (2.17) is rewritten as

$$\dot z = \mathscr{A} z + \mathscr{B} w \quad (2.18)$$

with obvious definitions of z, $\mathscr{A}$, and $\mathscr{B}$. Let Q^0 be the variance matrix of the initial conditions $z(0)$. The matrix Q^0 is partitioned, compatibly with the partitioning of z, as

$$Q^0 = \begin{bmatrix} Q_{11}^0 & Q_{12}^0 & Q_{13}^0 & Q_{14}^0 \\ & Q_{22}^0 & Q_{23}^0 & Q_{24}^0 \\ & & Q_{33}^0 & Q_{34}^0 \\ & & & Q_{44}^0 \end{bmatrix}.$$

By assumption, we have

$$Q_{ij}^0(\mu) = 0(\mu^N), \qquad \text{for } ij = 12, 14, 23, \text{ and } 34$$

and

$$Q_{ij}^0(\mu) = 0(\mu^{2N}), \qquad \text{for } ij = 22, 24, \text{ and } 44. \quad (2.19)$$

Let $Q(t)$ be the variance matrix of $z(t)$. It is well-known [14] that $Q(t)$ is the solution of the Lyapunov matrix differential equation

$$\dot Q = \mathscr{A} Q + Q\mathscr{A}^T + \mathscr{B} V \mathscr{B}^T, \qquad Q(0) = Q^0. \quad (2.20)$$

Because of the presence of $1/\mu$ terms in $\mathscr{A}$ and $\mathscr{B}$, Q is sought in the form

$$Q = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & Q_{14} \\ & Q_{22} & Q_{23} & Q_{24} \\ & & \frac{1}{\mu}Q_{33} & \frac{1}{\mu}Q_{34} \\ & & & \frac{1}{\mu}Q_{44} \end{bmatrix}. \quad (2.21)$$

Substituting (2.21) in (2.20) and partitioning the Lyapunov equation we get ten linear, time-invariant, ordinary differential equations seven of which are singularly perturbed. These are the equations involving the derivatives of Q_{13}, Q_{14}, Q_{23}, Q_{24}, Q_{33}, Q_{34}, and Q_{44}. The asymptotic behavior as $\mu \to 0$ over the interval $[0, \infty)$ can be studied using the method of [15]. Using [15] we get that for each small $\mu > 0$, there exists a unique solution $Q_{ij}(t, \mu)$ on $0 \leqslant t < \infty$ which is given by

$$Q_{ij}(t,\mu) = P_{ij}(t,\mu) + K_{ij}(\tau,\mu), \qquad \tau = t/\mu \quad (2.22)$$

where $P_{ij}(t, \mu)$ and $K_{ij}(\tau, \mu)$ are the outer and boundary-layer solutions defined in [15]. Moreover, the outer and boundary-layer solutions have Taylor expansions in μ such that for any integer $L > 0, Q_{ij}(t, \mu)$ is given by

$$Q_{ij}(t,\mu) = \sum_{r=0}^{L-1}\left[P_{ij}^{(r)}(t) + K_{ij}^{(r)}(\tau)\right]\mu^r + 0(\mu^L) \quad (2.23)$$

where $0(\mu^L)$ hold uniformly for $0 \leqslant t < \infty$. It can be shown (see [16] for detailed manipulations) that

$$P_{ij}^{(r)}(t) = 0 \quad \text{and} \quad K_{ij}^{(r)}(\tau) = 0, \qquad \forall r \in [0, N-1], \qquad \text{for } ij = 12, 14, 23, 34, \quad (2.24)$$

and

$$P_{ij}^{(r)}(t)=0 \quad \text{and} \quad K_{ij}^{(r)}(\tau)=0, \qquad \forall r\in[0,2N-1], \quad \text{for } ij=22,24,44. \tag{2.25}$$

Hence,

$$Q_{ij}(t,\mu)=0(\mu^N), \qquad \text{for } ij=12, 14, 23, \text{ and } 34, \tag{2.26}$$

and

$$Q_{ij}(t,\mu)=0(\mu^{2N}), \qquad \text{for } ij=22, 24, \text{ and } 44, \tag{2.27}$$

which proves (2.13)–(2.15). If (2.10) is not satisfied, the variances of e_x and e_y are evaluated only at steady state where the differential equations are replaced by their equilibrium algebraic equations. Repetition of the above argument shows that (2.13)–(2.15) hold at steady state.

Proof of Theorem 2: Since σ is evaluated at steady state, it is given by

$$\sigma=\operatorname{tr}\left[H^THQ̄_{11}+2H^TJ\bar{Q}_{13}^T\right]+\operatorname{tr}\left[\frac{1}{\mu}J^TJ\bar{Q}_{33}\right] \tag{2.28}$$

where

$$\bar{Q}_{ij}(\mu)=\lim_{t\to\infty}Q_{ij}(t,\mu)=\lim_{t\to\infty}P_{ij}(t,\mu) \tag{2.29}$$

because $K_{ij}(\tau,\mu)\to 0$ as $\tau\to\infty$ [15]. Using (2.23), we get, for any positive integer L, that

$$\bar{Q}_{ij}(\mu)=\sum_{r=0}^{L-1}\bar{Q}_{ij}^{(r)}\mu^r+0(\mu^L) \tag{2.30}$$

where $\bar{Q}_{ij}^{(r)}=\lim_{t\to\infty}P_{ij}^{(r)}(t)$. Since (2.23) holds for all $t\in[0,\infty)$, (2.26) and (2.27) hold for $\bar{Q}_{ij}$, as well. Now, σ_N is given by

$$\begin{aligned}\sigma_N=\operatorname{tr}\big[&H^{NT}H^N(\bar{Q}_{11}-2\bar{Q}_{12}+Q_{22})\\&+2H^{NT}J^N(\bar{Q}_{13}^T-\bar{Q}_{14}^T-\bar{Q}_{23}^T+\bar{Q}_{24}^T)\big]\\&+\frac{1}{\mu}\operatorname{tr}\left[J^{NT}J^N(\bar{Q}_{33}-2\bar{Q}_{34}+\bar{Q}_{44})\right].\end{aligned} \tag{2.31}$$

Subtracting (2.28) from (2.31) and using (2.26), (2.27) we get

$$\Delta\sigma=2\mu^{N-1}\operatorname{tr}\left[\left(J_N^N-J_N\right)^TJ_0\bar{Q}_{33}^{(0)}-J_0^TJ_0\bar{Q}_{34}^{(N)}\right]+0(\mu^N). \tag{2.32}$$

Recalling the partitioning of (2.20) it can be verified that $\bar{Q}_{33}^{(0)}$ and $\bar{Q}_{34}^{(N)}$ are, respectively, the unique solutions of the algebraic Lyapunov equations

$$D_0\bar{Q}_{33}^{(0)}+\bar{Q}_{33}^{(0)}D_0^T+F_0VF_0^T=0 \tag{2.33}$$

and

$$D_0\bar{Q}_{34}^{(N)}+\bar{Q}_{34}^{(N)}D_0^T+\bar{Q}_{33}^{(0)}(\Delta D)_N^T+F_0V(\Delta F)_N^T=0. \tag{2.34}$$

On the other hand,

$$\sigma=\frac{1}{\mu}\operatorname{tr}\left[J_0^TJ_0\bar{Q}_{33}^{(0)}\right]+0(1). \tag{2.35}$$

We want to show (2.16), i.e., $\Delta\sigma/\sigma=0(\mu^N)$. If $\operatorname{tr}[J_0^TJ_0\bar{Q}_{33}^{(0)}]\neq 0$, (2.16) is obtained by dividing (2.32) by (2.35). However, if $\operatorname{tr}[J_0^TJ_0\bar{Q}_{33}^{(0)}]=0$, such division will show only that $\Delta\sigma/\sigma=0(\mu^{N-1})$; further analysis is needed in this case to show (2.16).

Suppose that

$$\operatorname{tr}\left[J_0^TJ_0\bar{Q}_{33}^{(0)}\right]=0 \tag{2.36}$$

and recall that $\bar{Q}_{33}^{(0)}$ is given by

$$\bar{Q}_{33}^{(0)}=\int_0^\infty e^{D_0t}F_0VF_0^Te^{D_0^Tt}\,dt. \tag{2.37}$$

Substituting (2.37) in (2.36) yields

$$\int_0^\infty J_0e^{D_0t}F_0VF_0^Te^{D_0^Tt}J_0^T\,dt=0$$

which implies that

$$J_0e^{D_0t}F_0=0 \qquad \forall t. \tag{2.38}$$

Hence,

$$J_0\bar{Q}_{33}^{(0)}=0. \tag{2.39}$$

Recalling, that

$$\bar{Q}_{34}^{(N)}=\int_0^\infty e^{D_0t}\left[Q_{33}^{(0)}(\Delta D)_N^T+F_0V(\Delta F)_N^T\right]e^{D_0^Tt}\,dt,$$

and using (2.39), it can be shown that

$$J_0\bar{Q}_{34}^{(N)}=0. \tag{2.40}$$

The use of (2.39) and (2.40) in (2.32) shows that $\Delta\sigma=0(\mu^N)$ which completes the proof of Theorem 2.

III. Linear-Quadratic-Gaussian Control

Consider the singularly perturbed system

$$\dot{x}_1(t)=A_{11}x_1(t)+A_{12}x_2(t)+B_1u(t)+G_1w_1(t), \tag{3.1}$$

$$\mu\dot{x}_2(t)=A_{21}x_1(t)+A_{22}x_2(t)+B_2u(t)+G_2w_1(t), \tag{3.2}$$

$$y(t)=C_1x_1(t)+C_2x_2(t)+w_2(t) \tag{3.3}$$

with the performance criterion

$$J=\lim_{\substack{t_0\to-\infty\\t_1\to\infty}}\frac{1}{t_1-t_0}E\left\{\int_{t_0}^{t_1}\left[z^T(t)z(t)+u^T(t)Ru(t)\,dt\right]\right\}, \qquad R>0, \tag{3.4}$$

where $x_1\in R^{n_1}$, $x_2\in R^{n_2}$ comprise the state vector, $u\in R^m$ is the control input, $y\in R^l$ is the observed output, $w_1\in R^r$ and $w_2\in R^l$ are independent zero-mean stationary white Gaussian noise processes with intensities $V_1>0$ and $V_2>0$, respectively, and $z\in R^s$ is the controlled output which is given by

$$z(t)=D_1x_1(t)+D_2x_2(t).$$

The optimal control law is given by [10]

$$\begin{aligned}\dot{\hat{x}}_1(t)=&A_{11}\hat{x}_1(t)+A_{12}\hat{x}_2(t)+B_1u(t)\\&+K_1(\mu)[y(t)-C_1\hat{x}_1(t)-C_2\hat{x}_2(t)],\end{aligned} \tag{3.5}$$

$$\begin{aligned}\mu\dot{\hat{x}}_2(t)=&A_{21}\hat{x}_1(t)+A_{22}\hat{x}_2(t)+B_2u(t)\\&+K_2(\mu)[y(t)-C_1\hat{x}_1(t)-C_2\hat{x}_2(t)],\end{aligned} \tag{3.6}$$

$$u(t)=-[F_1(\mu)\hat{x}_1(t)+F_2(\mu)\hat{x}_2(t)]. \tag{3.7}$$

The regulator gains F_1 and F_2 are given by

$$F_1 = R^{-1}(B_1^T P_1 + B_2^T P_{12}^T), \qquad F_2 = R^{-1}(\mu B_1^T P_{12} + B_2^T P_2) \tag{3.8}$$

where P_1, P_{12}, and P_2 satisfy the algebraic equations

$$0 = P_1 A_{11} + P_{12} A_{21} + A_{11}^T P_1 + A_{21}^T P_{12}^T + D_1^T D_1 - (P_1 B_1 + P_{12} B_2) R^{-1} (B_1^T P_1 + B_2^T P_{12}^T), \tag{3.9}$$

$$0 = P_1 A_{12} + P_{12} A_{22} + \mu A_{11}^T P_{12} + A_{21}^T P_2 + D_1^T D_2 - (P_1 B_1 + P_{12} B_2) R^{-1} (\mu B_1^T P_{12} + B_2^T P_2), \tag{3.10}$$

$$0 = \mu P_{12}^T A_{12} + P_2 A_{22} + \mu A_{12}^T P_{12} + A_{22}^T P_2 + D_2^T D_2 - (\mu P_{12}^T B_1 + P_2 B_2) R^{-1} (\mu B_1^T P_{12} + B_2^T P_2) \tag{3.11}$$

while the filter gains K_1 and K_2 are given by

$$K_1 = (Q_1 C_1^T + Q_{12} C_2^T) V_2^{-1}, \qquad K_2 = (\mu Q_{12}^T C_1^T + Q_2 C_2^T) V_2^{-1} \tag{3.12}$$

where Q_1, Q_{12}, and Q_2 satisfy the algebraic equations

$$0 = A_{11} Q_1 + A_{12} Q_{12}^T + Q_1 A_{11}^T + Q_{12} A_{12}^T + G_1 V_1 G_1^T - (Q_1 C_1^T + Q_{12} C_2^T) V_2^{-1} (C_1 Q_1 + C_2 Q_{12}^T), \tag{3.13}$$

$$0 = \mu A_{11} Q_{12} + A_{12} Q_2 + Q_1 A_{21}^T + Q_{12} A_{22}^T + G_1 V_1 G_2^T - (Q_1 C_1^T + Q_{12} C_2^T) V_2^{-1} (\mu C_1 Q_{12} + C_2 Q_2), \tag{3.14}$$

$$0 = \mu A_{21} Q_{12} + A_{22} Q_2 + \mu Q_{12}^T A_{21}^T + Q_2 A_{22}^T + G_2 V_1 G_2^T - (\mu Q_{12}^T C_1^T + Q_2 C_2^T) V_2^{-1} (\mu C_1 Q_{12} + C_2 Q_2). \tag{3.15}$$

Equations (3.9)–(3.11) and (3.13)–(3.15) are dual and their solutions for small μ have been studied in [5], [8]. We recall some properties of the solutions which will be used later. Let us start with (3.13)–(3.15). Setting $\mu = 0$ in (3.13)–(3.15) decouples the equations in the following manner. First (3.15) takes the form

$$0 = A_{22} Q_2(0) + Q_2(0) A_{22}^T + G_2 V_1 G_2^T - Q_2(0) C_2^T V_2^{-1} C_2 Q_2(0) \tag{3.16}$$

which is a familiar algebraic Riccati equation. Assuming that the triple (A_{22}, G_2, C_2) is stabilizable-detectable guarantees [14] that (3.16) has a unique positive semidefinite solution such that

$$\operatorname{Re} \lambda (A_{22} - K_{22} C_2) < 0 \tag{3.17}$$

where $K_{22} \triangleq Q_{22}(0) C_2^T V_2^{-1} = K_2(0)$. Second, with $\mu = 0$ (3.14) is linear in $Q_{12}(0)$ and can be used to express $Q_{12}(0)$ in terms of $Q_1(0)$ and $Q_2(0)$. Third, eliminating $Q_{12}(0)$ from (3.13) and then using (3.15) to eliminate $Q_2(0)$, it has been shown in [8] that $Q_1(0)$ satisfies the algebraic Riccati equation

$$0 = [A_s - G_s V_1 H_s^T V_s^{-1} C_s] Q_1(0) + Q_1(0) [A_s - G_s V_1 H_s^T V_s^{-1} C_s]^T + G_s [V_1 - V_1 H_s^T V_s^{-1} H_s V_1] G_s^T - Q_1(0) C_s^T V_s^{-1} C_s Q_1(0), \tag{3.18}$$

where

$$A_s = A_{11} - A_{12} A_{22}^{-1} A_{21}, \qquad G_s = G_1 - A_{12} A_{22}^{-1} G_2,$$
$$H_s = -C_2 A_{22}^{-1} G_2,$$
$$V_s = V_2 + H_s V_1 H_s^T, \qquad C_s = C_1 - C_2 A_{22}^{-1} A_{21}.$$

Assuming that the triple (A_s, G_s, C_s) is stabilizable-detectable guarantees [14] that (3.12) has a unique positive semidefinite solution such that

$$\operatorname{Re} \lambda (A_s - K_s C_s) < 0 \tag{3.19}$$

where $K_s \triangleq [Q_1(0) C_s^T + G_s V_1 H_s] V_s^{-1}$. Based on the stability properties (3.17) and (3.19) and using implicit function theorem arguments the following lemma was proved in [8].

Lemma 1: If A_{22} is nonsingular and the triples (A_s, G_s, C_s) and (A_{22}, G_2, C_2) are stabilizable-detectable, then for sufficiently small μ (3.7)–(3.9) have a unique stabilizing solution which possesses a power series expansion at $\mu = 0$.

The solution of (3.9)–(3.11) has dual properties. It is seen that if (A_{22}, B_2, D_2) is stabilizable-detectable, then $P_2(0)$ is the unique positive semidefinite solution of the algebraic Riccati equation

$$0 = P_2(0) A_{22} + A_{22}^T P_2(0) + D_2^T D_2 - P_2(0) B_2 R^{-1} B_2^T P_2(0) \tag{3.20}$$

and

$$\operatorname{Re} \lambda (A_{22} - B_2 F_{22}) < 0 \tag{3.21}$$

where

$$F_{22} \triangleq R^{-1} B_2^T P_2(0) = F_2(0).$$

Also, if (A_s, B_s, D_s) is stabilizable-detectable, then $P_1(0)$ is the unique positive semidefinite solution of the algebraic Riccati equation

$$0 = P_1(0) [A_s - B_s R_s^{-1} E_s^T D_s] + [A_s - B_s R_s^{-1} E_s^T D_s]^T P_1(0) + D_s^T [I - E_s R_s^{-1} E_s] D_s - P_1(0) B_s R_s^{-1} B_s^T P_1(0), \tag{3.22}$$

and

$$\operatorname{Re} \lambda (A_s - B_s F_s) < 0, \tag{3.23}$$

where

$$B_s = B_1 - A_{12} A_{22}^{-1} B_2, \qquad D_s = D_1 - D_2 A_{22}^{-1} A_{21},$$
$$E_s = -D_2 A_{22}^{-1} B_2,$$
$$R_s = R + E_s^T E_s, \qquad F_s = R_s^{-1}(E_s^T D_s + B_s^T P_s).$$

The existence and uniqueness of the solution of (3.9)–(3.11) is established in the following lemma [8] which is the dual of Lemma 1.

Lemma 2: If A_{22} is nonsingular and the triples (A_s, B_s, D_s) and (A_{22}, B_2, D_2) are stabilizable-detectable, then for sufficiently small μ (3.9)–(3.11) have a unique stabilizing solution which possess a power series expansion at $\mu = 0$.

Previous attempts to simplify the optimal control exploiting the two-time scale nature of the system are due to Haddad and Kokotovic [10] and Teneketzis and Sandell [11]. A special case of [10] was studied by Khalil [18]. In [10], an approximate control law has been derived through slow–fast decompositions of the singularly perturbed system. The control law is shown to be near-optimal by showing, via involved study of J, that $\Delta J/J$ is $0(\mu)$ where ΔJ is the increase in the performance criterion. In [11], an approximate control law has been derived from the optimal one (3.5)–(3.7) by, first, substituting u from (3.7) into (3.5) and (3.6), second, formally setting $\mu = 0$ in the left-hand side of (3.6) and, finally, using the resulting algebraic equation to eliminate $\hat{x}_2$ from (3.5). It has been shown that the error, between the state of the system under optimal control and the state of the system under the approximate control, converges to zero, in the mean square sense, as $\mu \to 0$. A common feature in both approaches is that the decomposition of the Kalman filter into slow-mode and fast-mode filters is achieved via formal elimination of the fast variables from the slow equations. Because of the formal nature of deriving approximations, there is no way of modifying the approximate controls in order to get higher order approximations.

Our approach toward simplifying the LQG regulator separates the decomposition and approximation tasks. To decompose the Kalman filter into slow-mode and fast-mode filters there is no need to introduce any approximations. All that is needed is to represent the Kalman filter in new coordinates where the filter state variables cluster into n_1 slow variables and n_2 fast variables. We achieve this by using a decoupling state transformation that was introduced by Chang [12] to block diagonalize singularly perturbed systems. In fact because our system is time-invariant we only need to use the special case of the transformation that was introduced by Kokotovic [17] and which uses algebraic equations rather than differential equations as in the general form of Chang. The basic properties of the transformation are summarized in the Appendix. Since the objective of the decomposition task is to decouple the slow and fast variables of the Kalman filter, we have to examine the power spectra of the filter variables. To characterize the power spectra of the state variables of the Kalman filter we view the filter as a system driven by the innovation process, i.e.,

$$\dot{\hat{x}}_1(t) = (A_{11} - B_1F_1)\hat{x}_1(t) + (A_{12} - B_1F_2)\hat{x}_2(t) + K_1v(t) \quad (3.24)$$

$$\mu\dot{\hat{x}}_2(t) = (A_{21} - B_2F_1)\hat{x}_1(t) + (A_{22} - B_2F_2)\hat{x}_2(t) + K_2v(t). \quad (3.25)$$

Since the innovation process $v(t)$ is white noise the power spectrum of $\hat{x}(t)$ is determined by the homogeneous part of (3.24), (3.25) which is a well-defined singularly perturbed system; recall that, by Lemma 2, $(A_{22} - B_2F_2)$ is nonsingular at $\mu = 0$. Therefore, we block diagonalize the homogeneous part of (3.24), (3.25) using the transformation

$$\begin{bmatrix} \hat{\eta}_1(t) \\ \hat{\eta}_2(t) \end{bmatrix} = \begin{bmatrix} I_{n_1} - \mu M_1L_1 & -\mu M_1 \\ L_1 & I_{n_2} \end{bmatrix} \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} \quad (3.26)$$

where the matrices M_1 and L_1 satisfy equations similar to (A4) and (A8) of the Appendix with $(A_{ij} - B_iF_j)$ replacing A_{ij}. The optimal feedback control, expressed in the new coordinates, is given by

$$\dot{\hat{\eta}}_1(t) = [(A_{11} - B_1F_1) - (A_{12} - B_1F_2)L_1]\hat{\eta}_1(t) + (K_1 - M_1K_2 - \mu M_1L_1K_1)v(t) \quad (3.27)$$

$$\mu\dot{\hat{\eta}}_2(t) = [(A_{22} - B_2F_2) + \mu L_1(A_{12} - B_1F_2)]\hat{\eta}_2(t) + (K_2 + \mu L_1K_1)v(t) \quad (3.28)$$

$$v(t) = y(t) - (C_1 - C_2L_1)\hat{\eta}_1(t) - [C_2 + \mu(C_1 - C_2L_1)M_1]\hat{\eta}_2(t) \quad (3.29)$$

$$u(t) = -(F_1 - F_2L_1)\hat{\eta}_1(t) - [F_2 + \mu(F_1 - F_2L_1)M_1]\hat{\eta}_2(t). \quad (3.30)$$

Equation (3.27) is a slow-mode filter whose state $\hat{\eta}_1$ has a frequency band of order one, while (3.28) is a fast-mode filter whose state $\hat{\eta}_2$ has a frequency band of order $1/\mu$. Implementation of the optimal feedback control as given by (3.27)–(3.30) should lead to a reduction in the on-line computations. In particular, in implementing (3.27) and (3.28) using numerical integration routines, different integration step sizes can be used. It is crucial that the decomposition of the Kalman filter is achieved without sacrificing the nice analytical properties of the system because the matrices L_1 and M_1 are analytic in μ. This makes the approximation task, to be discussed next, a feasible one.

Approximate control laws are defined by perturbing the right-hand side coefficients of (3.27)–(3.30). Specifically, the matrices F_1, F_2, K_1, K_2, M_1, and L_1 are approximated by their Nth-order approximations F_1^N, F_2^N, K_1^N, K_2^N, M_1^N, and L_1^N, respectively, where an Nth-order approximation of a matrix consists of the leading N terms of the expansion of that matrix. The state variables of the perturbed filter will be denoted by $\hat{\eta}_1^N$ and $\hat{\eta}_2^N$. If the filter (3.5), (3.6) in the optimal control law are initiated at certain initial conditions $\hat{x}_1(0)$ and $\hat{x}_2(0)$, then the initial conditions $\hat{\eta}_1^N(0)$ and $\hat{\eta}_2^N(0)$ should be taken as

$$\hat{\eta}_1^N(0) = \left(I_{n_1} - \mu M_1^NL_1^N\right)\hat{x}_1(0) - \mu M_1^N\hat{x}_2(0) \quad (3.31)$$

$$\hat{\eta}_2^N(0) = L_1^N\hat{x}_1(0) + \hat{x}_2(0). \quad (3.32)$$

The near-optimality of the proposed control law is established in the following theorem.

Theorem 3: Suppose that the conditions of Lemma 1 and Lemma 2 hold. Let $x_1(t)$ and $x_2(t)$ be the optimal trajectories and J be the optimal value of the performance criterion. Let $\bar{x}_1(t)$, $\bar{x}_2(t)$, and $\bar{J}$ be the corresponding quantities under the Nth-order approximate control law and let $\Delta J = \bar{J} - J$. Then

$$\frac{\Delta J}{J} = 0(\mu^N), \quad (3.33)$$

$$\operatorname{var}(x_1(t) - \bar{x}_1(t)) = 0(\mu^{2N}), \qquad (\text{as } t \to \infty), \quad (3.34)$$

and

$$\operatorname{var}(x_2(t) - \bar{x}_2(t)) = 0(\mu^{2N-1}), \qquad (\text{as } t \to \infty). \quad (3.35)$$

Moreover, if $\hat{\eta}_1(0)$ and $\hat{\eta}_2(0)$ are chosen according to (3.31), (3.32), then (3.34) and (3.35) hold for all $t \geq 0$.

Proof of Theorem 3: The result of Section II is employed by studying systems of equations driven by white noise. For the optimal control consider the equations

$$\begin{bmatrix} \dot{\hat{\eta}}_1 \\ \dot{e}_1 \\ \mu\dot{\hat{\eta}}_2 \\ \mu\dot{e}_2 \end{bmatrix} = \begin{bmatrix} \mathscr{A}_{11} & \mathscr{A}_{12} \\ \mathscr{A}_{21} & \mathscr{A}_{22} \end{bmatrix} \begin{bmatrix} \hat{\eta}_1 \\ e_1 \\ \hat{\eta}_2 \\ e_2 \end{bmatrix} + \begin{bmatrix} \mathscr{B}_1 \\ \mathscr{B}_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \quad (3.36)$$

where $e_1 = \eta_1 - \hat{\eta}_1$ and $e_2 = \eta_2 - \hat{\eta}_2$ are the estimation errors, and η_1, η_2 are transformations of x_1, x_2 using (3.26). The corresponding equation for the approximate control is

$$\begin{bmatrix} \dot{\hat{\eta}}_1^N \\ \dot{e}_1^N \\ \mu\dot{\hat{\eta}}_2^N \\ \mu\dot{e}_2^N \end{bmatrix} = \begin{bmatrix} \mathscr{A}_{11}^N & \mathscr{A}_{12}^N \\ \mathscr{A}_{21}^N & \mathscr{A}_{22}^N \end{bmatrix} \begin{bmatrix} \hat{\eta}_1^N \\ e_1^N \\ \hat{\eta}_2^N \\ e_2^N \end{bmatrix} + \begin{bmatrix} \mathscr{B}_1^N \\ \mathscr{B}_2^N \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \quad (3.37)$$

where $e_1^N = \eta_1 - \hat{\eta}_1^N$ and $e_2^N = \eta_2 - \hat{\eta}_2^N$ are the estimation errors. The matrices $\mathscr{A}_{ij}$, $\mathscr{B}_i$ and $\mathscr{A}_{ij}^N, \mathscr{B}_i^N$ in (3.36) and (3.37), respectively, are obtained in an obvious way. It can be verified that

$$\mathscr{A}_{ij} - \mathscr{A}_{ij}^N = 0(\mu^N) \quad \text{and} \quad \mathscr{B}_i - \mathscr{B}_i^N = 0(\mu^N)$$

so that condition (2.9) is satisfied. To apply Theorem 1 we should verify conditions (2.5) and (2.6), i.e., we should show that $\mathscr{A}_{22}(0)$ and $[\mathscr{A}_{11}(0)-\mathscr{A}_{12}(0)\mathscr{A}_{22}^{-1}(0)\mathscr{A}_{21}(0)]$ are Hurwitz matrices. It can be shown that

$$\mathscr{A}_{22}(0)=\begin{bmatrix} A_{22}-B_{22}F_{22} & K_{22}C_2 \\ 0 & A_{22}-K_{22}C_2 \end{bmatrix}, \tag{3.38}$$

$$\mathscr{A}_{11}(0)-\mathscr{A}_{12}(0)\mathscr{A}_{22}^{-1}(0)\mathscr{A}_{21}(0)=\begin{bmatrix} A_s-B_sF_s & \pi_{12} \\ 0 & A_s-K_sC_s \end{bmatrix}. \tag{3.39}$$

Stability of $\mathscr{A}_{22}(0)$ and $[\mathscr{A}_{11}(0)-\mathscr{A}_{12}(0)\mathscr{A}_{22}^{-1}(0)\mathscr{A}_{21}(0)]$ results from the stability properties (3.17), (3.19), (3.21), and (3.23). The use of Theorem 1 proves (3.34) and (3.35). Theorem 2 leads to (3.33) after verifying that when J and $\bar{J}$ are expressed as quadratic forms in $(\hat{\eta}_i, e_i)$ and $(\hat{\eta}_i^N, e_i^N)$, respectively, the matrices in $\bar{J}$ are $0(\mu^N)$ perturbations of the matrices in J.

A special case of interest is the case when μ is small enough that $0(\mu)$ suboptimality is acceptable. In this case a simplified first-order near-optimal control law is defined by neglecting all $0(\mu)$ coefficients on the right-hand side of (3.27)–(3.30). This results, after some algebraic manipulations, in

$$\dot{\hat{\eta}}_1(t)=(A_s-B_sF_s)\hat{\eta}_1(t)+\left[K_s(I-C_2A_{22}^{-1}K_{22})+B_sF_{22}(A_{22}-B_2F_{22})^{-1}K_{22}\right]v(t) \tag{3.40}$$

$$\mu\dot{\hat{\eta}}_2(t)=(A_{22}-B_2F_{22})\hat{\eta}_2(t)+K_{22}v(t) \tag{3.41}$$

$$v(t)=y(t)-(C_s-N_sF_s)\hat{\eta}_1(t)-C_2\hat{\eta}_2(t) \tag{3.42}$$

$$u(t)=-F_s\hat{\eta}_1(t)-F_{22}\hat{\eta}_2(t) \tag{3.43}$$

where $N_s=-C_2A_{22}^{-1}B_2$. Similar to the proof of Theorem 3, it can be verified that the use of the control law (3.40)–(3.43) is near-optimal. Theorem 4, which is given without proof, summarizes this conclusion.

Theorem 4: If the conditions of Lemmas 1 and 2 are satisfied, then

$$\frac{\Delta J}{J}=0(\mu), \tag{3.44}$$

$$\operatorname{var}(x_1(t)-\bar{x}_1(t))=0(\mu^2), \tag{3.45}$$

and

$$\operatorname{var}(x_2(t)-\bar{x}_2(t))=0(\mu) \tag{3.46}$$

where (3.45) and (3.46) hold at steady state for any choice of $\hat{\eta}_1(0)$ and $\hat{\eta}_2(0)$, and hold for all $t\geqslant 0$ if $\hat{\eta}_1(0)$ and $\hat{\eta}_2(0)$ are chosen as $\hat{\eta}_1(0)=\hat{x}_1(0)$ and $\hat{\eta}_2(0)=\hat{x}_2(0)+A_{22}^{-1}(A_{21}-B_2F_s)\hat{x}_1(0)$.

To derive the near-optimal control law (3.40)–(3.43) one need not consider the overall LQG problem. Rather, two lower order LQG problems defined by Haddad and Kokotovic [10] are solved under the conditions of Lemmas 1 and 2. The first one is a slow LQG problem defined by

$$\dot{x}_s(t)=A_sx_s(t)+B_su_s(t)+G_sw_1(t), \tag{3.47}$$

$$y(t)=C_sx_s(t)+N_su_s(t)+H_sw_1(t)+w_2(t), \tag{3.48}$$

$$J_s=\lim_{\substack{t_0\to-\infty\\ t_1\to\infty}}\frac{1}{t_1-t_0}\cdot E\left\{\int_{t_0}^{t_1}\left[x_s^TD_s^TD_sx_s+2x_s^TE_s^TD_su_s+u_s^TR_su_s\,dt\right]\right\} \tag{3.49}$$

and its optimal solution is given by

$$u_s=-F_s\hat{x}_s \tag{3.50}$$

where the slow filter for any given u satisfies

$$\dot{\hat{x}}_s(t)=A_s\hat{x}_s(t)+B_su(t)+K_s[y(t)-N_su(t)-C_s\hat{x}_s(t)]. \tag{3.51}$$

The second problem is a fast LQG problem defined by

$$\mu\dot{x}_f(t)=A_{22}x_f(t)+B_2u_f(t)+G_2w_1(t) \tag{3.52}$$

$$y_f(t)=C_2x_f(t)+w_2(t) \tag{3.53}$$

$$J_f=\lim_{\substack{t_0\to-\infty\\ t_1\to\infty}}\frac{1}{t_1-t_0}E\left\{\int_{t_0}^{t_1}\left[x_f^TD_2^TD_2x_f+u_f^TRu_f\right]dt\right\} \tag{3.54}$$

and its optimal solution is given by

$$u_f(t)=-F_{22}\hat{x}_f(t) \tag{3.55}$$

$$\mu\dot{\hat{x}}_f(t)=A_{22}\hat{x}_f(t)+B_2u_f(t)+K_{22}(y_f(t)-C_2\hat{x}_f(t)). \tag{3.56}$$

Once F_s, K_s, F_{22}, and K_{22} are computed, the simplified first-order near-optimal control law (3.40)–(3.43) can be implemented. The resemblance with [10] goes far beyond just computing the coefficients. In [10] a near-optimal control law is taken as

$$u(t)=u_s+u_f=-F_s\hat{x}_s(t)-F_{22}\hat{x}_f(t). \tag{3.57}$$

Comparing (3.57) to (3.43) shows that we should expect $\hat{x}_s$ and $\hat{x}_f$ to be equivalent, in some sense, to $\hat{\eta}_1$ and $\hat{\eta}_2$. In [10] the fast filter (3.56) is implemented by taking y_f as $y_f=y-C_s\hat{x}_s+N_sF_s\hat{x}_s$ so that the fast filter can be written as

$$\mu\dot{\hat{x}}_f(t)=(A_{22}-B_2F_{22})\hat{x}_f(t)+K_{22}v \tag{3.58}$$

where

$$v(t)=y(t)-C_s\hat{x}_s(t)+N_sF_s\hat{x}_s(t)-C_2\hat{x}_f(t). \tag{3.59}$$

Comparing (3.58), (3.59) to (3.41), (3.42) shows that the two fast filters are indeed equivalent. To see the equivalence of the slow filters we rewrite (3.51) as

$$\dot{\hat{x}}_s(t)=(A_s-B_sF_s)\hat{x}_s(t)-B_sF_{22}\hat{x}_f(t)+K_s\left[v(t)+N_sF_{22}\hat{x}_f(t)+C_2\hat{x}_f(t)\right]. \tag{3.60}$$

Following the arguments of [10] one can say that, as an input to the slow filter (3.60), $\hat{x}_f(t)$ can be approximated by

$$\hat{x}_f(t)=-(A_{22}-B_2F_{22})^{-1}K_{22}v(t). \tag{3.61}$$

If the formal expression (3.61) is substituted in (3.60), it can be shown that $\hat{x}_s(t)$ satisfies the same slow filter equation (3.40) as $\hat{\eta}_1(t)$ with v as the deriving term. This shows that the simplified first-order near-optimal control (3.40)–(3.43) is indeed equivalent to the near-optimal control of Haddad and Kokotovic [10].

IV. A Numerical Example

In order to demonstrate the numerical behavior of the near-optimum design of singularly perturbed LQG regulators, we present results for an LQG controller of an F-8 aircraft which was considered in [11]. The system model is given by

$$\begin{bmatrix}\dot{\xi}_1\\ \dot{\xi}_2\\ \dot{\xi}_3\\ \dot{\xi}_4\end{bmatrix}=\begin{bmatrix}-1.357\times10^{-2} & -32.2 & -46.3 & 0\\ 1.2\times10^{-4} & 0 & 1.214 & 0\\ -1.212\times10^{-4} & 0 & -1.214 & 1\\ 5.7\times10^{-4} & 0 & -9.01 & -0.6696\end{bmatrix}\begin{bmatrix}\xi_1\\ \xi_2\\ \xi_3\\ \xi_4\end{bmatrix}+\begin{bmatrix}-0.433\\ 0.1394\\ -0.1394\\ -0.1577\end{bmatrix}u+\begin{bmatrix}-46.3\\ 1.214\\ -1.214\\ -9.01\end{bmatrix}w_1 \tag{4.1}$$

$$y=\begin{bmatrix}0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\xi_1\\ \xi_2\\ \xi_3\\ \xi_4\end{bmatrix}+w_2 \tag{4.2}$$

where the white noise processes w_1 and w_2 are independent and have intensities $V_1=3.15\times10^{-4}$ and $V_2=\mathrm{diag}[6.859\times10^{-4},40]$. The performance criterion is

$$J=\lim_{\substack{t_0\to-\infty\\ t_f\to\infty}}\frac{1}{t_f-t_0}E\int_{t_0}^{t_f}\left[0.01\xi_1^2+3260\left(\xi_3^2+\xi_4^2+u^2\right)\right]dt. \tag{4.3}$$

The reader is referred to [11] for discussion of the modeling aspects and the choice of J.

The open-loop eigenvalues are $-0.94\pm j2.98$ and $-0.0075\pm j0.076$ which shows clearly the two-time-scale property of the system. The choice of state variables adopted in [11] led nicely to a formulation in which the first two variables are slow variables. A logical choice of the parameter μ is $\mu=0.025$ which is roughly the ratio of the magnitude of the slow eigenvalues to the magnitude of the fast eigenvalues. The singularly perturbed nature of this system becomes more evident [i.e., the right-hand side coefficients of the last two rows of the state equations are of $0(1/\mu)$, and those of the first two rows are $0(1)$] by scaling the variables as follows: $\tilde{\xi}=\mathrm{diag}(1,500,15000,5000)\,\xi$, $\tilde{u}=1000u$, $\tilde{w}_1=100w_1$, $\tilde{y}=\mathrm{diag}(100,1)y$ and $\tilde{w}_2=\mathrm{diag}(100,1)w_2$. Introducing μ artificially by multiplying the left-hand sides of the last two state equations by μ and the right-hand sides by 0.025, the system takes the singularly perturbed form (3.1)–(3.4) with

$$A_{11}=\begin{bmatrix}-0.01357 & -0.0644\\ 0.06 & 0\end{bmatrix},\quad A_{12}=\begin{bmatrix}-0.003087 & 0\\ 0.040467 & 0\end{bmatrix},$$

$$A_{21}=\begin{bmatrix}-0.045375 & 0\\ 0.07125 & 0\end{bmatrix},\quad A_{22}=\begin{bmatrix}-0.03035 & 0.075\\ -0.075083 & -0.01674\end{bmatrix},$$

$$B_1=\begin{bmatrix}-0.000433\\ 0.0697\end{bmatrix},\quad B_2=\begin{bmatrix}-0.052275\\ 0.019712\end{bmatrix},$$

$$G_1=\begin{bmatrix}-0.463\\ 6.07\end{bmatrix},\quad G_2=\begin{bmatrix}-4.5525\\ -11.262499\end{bmatrix},$$

$$C_1=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix},\quad C_2=\begin{bmatrix}0 & 0.02\\ 0 & 0\end{bmatrix},\quad R=3.26\times10^{-3}$$

$$D_1^T=\begin{bmatrix}0.1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix},$$

$$D_2^T=\begin{bmatrix}0 & 0 & 3.81\times10^{-3} & 0\\ 0 & 0 & 0 & 11.42\times10^{-3}\end{bmatrix}$$

$$V_1=3.15\quad V_2=\mathrm{diag}\{6.859,40\}.$$

Componentwise results for F_1^N, F_2^N, K_1^N, K_2^N, L^N, and M^N approximations, for $N=1,2,3$, and corresponding values of the performance criterion are shown in Table I. Shown also are the simplified first-order approximation and optimal solutions. The rate of convergence of the coefficients and $\Delta J/J$ towards the optimal solution can be noticed. In this example the simplified first-order approximation behaves better than the first-order one. But, in general, it could be the other way around.

TABLE I
AN F-8 AIRCRAFT LQG CONTROLLER EXAMPLE: $\mu=0.025$

	SIMPLIFIED	FIRST	SECOND	THIRD	OPTIMAL
F_{11}	-0.435272		-0.435152	-0.435318	-0.435326
F_{12}	1.092964		1.109812	1.109874	1.109888
F_{21}	-0.019812		-0.016716	-0.016584	-0.016576
F_{22}	-0.002611		0.011683	0.011866	0.011873
K_{11}	0.231563		0.218580	0.219336	0.219508
K_{12}	0.000000		0.002134	0.003945	0.004146
K_{13}	-3.035830		-3.038478	-3.040135	-3.040065
K_{14}	0.000000		-0.027972	-0.028639	-0.028690
K_{21}	2.276873		2.278865	2.280198	2.280667
K_{22}	0.000000		0.031129	0.049660	0.049344
K_{23}	5.632790		5.634822	5.632759	5.632612
K_{24}	0.000000		0.049634	0.046851	0.047050
L_1	-0.574315		-0.604521	-0.605300	-0.605345
L_2	-0.416417		-0.436575	-0.436673	-0.436679
L_3	-1.150816		-1.163621	-1.164267	-1.164268
L_4	0.588607		0.607638	0.608499	0.608514
M_1	0.008400		0.017795	0.017854	0.017854
M_2	0.037518		0.031582	0.031527	0.031515
M_3	-0.111529		-0.111386	-0.111803	-0.111810
M_4	-0.508085		-0.512595	-0.512817	-0.512825
J	25.066690	25.141589	25.066652	25.066596	25.066595
$\frac{\Delta J}{J}$ %	3.79×10^{-4}	2.99×10^{-1}	2.274×10^{-4}	4×10^{-6}	-----

In order to illustrate the numerical behavior of the near-optimum solution as $\mu\to0$, μ is varied over the range [0.0025,0.2] and the percentage relative error in J is given in Table II; while doing that all the matrices A_{ij}, B_i, G_i, C_i, D_i, and R are kept fixed, i.e., only the solution for $\mu=0.025$ is meaningful as far as the F-8 aircraft controller is concerned. Table II verifies that $\Delta J^N/J$ is indeed $0(\mu^N)$ as $\mu\to0$.

V. DISCUSSIONS

The significance of the near-optimal control laws and Theorems 3 and 4 that have been derived in Section III is extending to the LQG problem the approximation procedures of the LQ problem. The Nth-order near-optimal control extends a corresponding result due to Yackel and Kokotovic [5] while the simplified first-order near-optimal control extends the composite control result of Chow and Kokotovic [8]. There are, however, important differences between approximating LQ and approximating LQG solutions. These differences are summarized in Table III and explained here in some detail. The first difference is not related to the approximation scheme, rather to the problem definition itself. While in the LQ problem the optimal value of J is $0(1)$, in the LQG problem the optimal value of J is, in general, $0(1/\mu)$, which results from having a white noise input multiplied by $1/\mu$. The LQG problem definition may be altered to avoid $0(1/\mu)$ optimal value of J; more about this point later. The second difference has to do with using Nth-order approximations. In the LQ problem it has been shown ([3]; see also [19]) that the use of Nth-order approximations in feedback coefficients results in $2N$th-order approximation in J, i.e., $(\Delta J/J)=0(\mu^{2N})$. In the LQG problem, Theorem 3 assured only that

TABLE II
BEHAVIOR OF RELATIVE ERROR AS $\mu \to 0$

μ	$\frac{\Delta J^{(1)}}{J}$ %	$\frac{\Delta J^{(2)}}{J}$ %	$\frac{\Delta J^{(3)}}{J}$ %
0.0025	0.002977	$<10^{-6}$	$<10^{-6}$
0.005	0.011914	$<10^{-6}$	$<10^{-6}$
0.01	0.0477	3.2×10^{-6}	$<10^{-6}$
0.025	0.299179	2.274×10^{-4}	4×10^{-6}
0.05	1.207573	5.3206×10^{-3}	1.436×10^{-4}
0.1	5.000657	0.134547	0.011531
0.2	24.299446	8.717748	4.455874

TABLE III
COMPARISON BETWEEN LQ AND LQG

		LQ	LQG
1	Optimal Value J	$0(1)$	$0(\frac{1}{\mu})$
2	N-th order approximation	$\frac{\Delta J}{J} = 0(\mu^{2N})$	$\frac{\Delta J}{J} = 0(\mu^{N})$
3	Simplified first order approximation	$\frac{\Delta J}{J} = 0(\mu^{2})$	$\frac{\Delta J}{J} = 0(\mu)$
4	Closeness of composite control response $(\bar{x}_1, \bar{x}_2)$ to optimal response (x_1, x_2)	$x_1 - \bar{x}_1 = 0(\mu)$ $x_2 - \bar{x}_2 = 0(\mu)$	$x_1 - \bar{x}_1 = 0(\mu)$ $x_2 - x_2 = 0(\mu^{1/2})$
5	Closeness of composite control response $(\bar{x}_1, \bar{x}_2)$ to slow and fast responses	$x_1 - x_s = 0(\mu)$ $x_2 - (x_f - A_{22}^{-1}(A_{21} - B_sF_s)x_s) = 0(\mu)$	Not asympotically close
6	Reduced control	$\frac{\Delta J}{J} = 0(\mu)$	Not near-optimal

$(\Delta J/J) = 0(\mu^N)$. The next three differences compare the simplified first-order near-optimal control law of the LQG problem to the composite control law of the LQ problem [8]. First, the use of the composite control in the LQ problem results in $(\Delta J/J) = 0(\mu^2)$, versus $(\Delta J/J) = 0(\mu)$ in the LQG problem. Second, if x_1 and x_2 are the states of the system under optimal control, and $\bar{x}_1$ and $\bar{x}_2$ are the states under near-optimal control, then in the LQ problem $x_1 - \bar{x}_1 = 0(\mu)$ and $x_2 - \bar{x}_2 = 0(\mu)$, while in the LQG problem $x_1 - \bar{x}_1 = 0(\mu)$ and $x_2 - \bar{x}_2 = 0(\mu^{1/2})$ in mean-square sense. Third, and that is an important difference, in the case of LQ problem it has been shown that $\bar{x}_1$ and $\bar{x}_2$ are asymptotically close to x_s and $x_f - A_{22}^{-1}(A_{21} - B_sF_s)x_s$ where x_s and x_f are the optimal trajectories of the slow and fast optimal control problems [8]. There is no corresponding result in the LQG problem, i.e., $\bar{x}_1$ is not asymptotically close to x_s [the solution of the slow optimal control problem (3.47)–(3.49)]. To see this notice that the asymptotic behavior of η_1 or x_1 is determined by the matrix $[\mathscr{A}_{11}(0) - \mathscr{A}_{12}(0)\mathscr{A}_{22}^{-1}(0)\mathscr{A}_{21}(0)]$ [which is given by (3.39)]. If a corresponding matrix for the optimal closed-loop slow problem is written down it will be exactly the same as (3.39) except that the upper right block π_{12} is, in general, not equal to that of $[\mathscr{A}_{11}(0) - \mathscr{A}_{12}(0)\mathscr{A}_{22}^{-1}(0)\mathscr{A}_{21}(0)]$. The fact that the on-diagonal blocks are the same and the matrix is block triangular has been employed in the proof of Theorem 5 to conclude the stability of $[\mathscr{A}_{11}(0) - \mathscr{A}_{12}(0)\mathscr{A}_{22}^{-1}(0)\mathscr{A}_{21}(0)]$ from the properties of the optimal slow solution. However, closeness of x_1 and x_s is not true in general. It is interesting to mention that the same observation was made by O'Reilly [20] while studying deterministic output feedback of singularly perturbed systems using controller-observer compensators which is the deterministic version of the LQG problem. Finally, the last difference is in the use of reduced controls. In the LQ problem it is known [8] that an approximate control law derived by solving only the slow optimal control problem, i.e., $u = -F_sx_1$ is near-optimal with $(\Delta J/J) = 0(\mu)$. This fact justifies using the slow reduced-order model as a basis for solving the LQ problem. In the LQG problem a similar reduced control will not be near-optimal. This results from the fact that in the LQG problem the value of J as $\mu \to 0$ is determined by both the slow and fast variables. In fact, when $J = 0(1/\mu)$, it is determined mainly by the fast variables. So, optimization of the fast variables is necessary to achieve near optimality. There are, however, special cases when the reduced control is near-optimal, e.g., when $D_2 = 0$ [10] or $G_2 = 0$ [18]. Among the six differences discussed above, the last two are important because they show features of the LQ problem that are not extendable to the LQG problem.

As we have pointed out, the classical formulation of the LQG problem which has been adopted in this paper leads to an optimal value of J which is $0(1/\mu)$. We have shown that despite this fact meaningful approximations can be defined and used to approximate the optimal value of J. The problem of divergent performance criteria can be avoided by altering the problem formulation. Khalil, Haddad, and Blankenship [21] studied an alternative LQG problem formulation in which various scaling parameters have been introduced in such a way that the value of J remains finite as $\mu \to 0$. For example, the coefficient of the white noise input in the singularly perturbed equations (3.2) is taken as $\mu^\alpha G_2$, $\alpha \geq 1/2$, instead of G_2 in the classical formulation. Such modifications guarantee that J will be well-defined as $\mu \to 0$, and they are extremely useful in dealing with nonlinear problems [22]. They have also been used in stochastic Nash games [23]. The modified LQG problem has been treated in [21] using the slow–fast decomposition approach of [10]. Naturally, the approxiamtion procedures of this paper can be applied to the modified LQG problem and the fundamental results of Section II, about approximations of singularly perturbed systems driven by white noise, will play the same essential role they have played in this paper. So far, there has not been an adequate comparative study of the modified LQG problem formulation versus the classical one. More work is needed in that direction.

Approximation methods for the estimation problem when $u = 0$ can be obtained in a way similar to those of the control problem. In fact setting the regulator gain F equal to zero in the Kalman filter equations of Section III results in the near-optimal Kalman filter in the open-loop case. Assuming that A_s and A_{22} are Hurwitz matrices theorems similar to Theorems 3 and 4 can be proved to assure that $\hat{x}_1$ and $\hat{x}_2$, the approximate estimates, are near-optimal in the sense that $\mathrm{var}(\hat{x}_1 - \hat{x}_1) = 0(\mu^N)$ and $\mathrm{var}(\hat{x}_2 - \hat{x}_2) = 0(\mu^{N-(1/2)})$ for Nth-order approximations.

Finally, we conclude our discussions by reexamining the decoupling transformation which was used to restructure the Kalman filter in Section III. There we used a transformation of the form

$$T = \begin{bmatrix} I - \mu ML & -\mu M \\ L & I \end{bmatrix} \tag{5.1}$$

with the specific choices $L = L_1$ and $M = M_1$. Since the purpose of the transformation is to represent the Kalman filter in new coordinates where the slow and fast variables are decoupled, one might principally examine many transformations which could achieve this goal. For example, appropriately selected modal transformations would certainly decouple slow and fast variables. However, we restricted our attention to the class of transformations represented by (5.1) in which L and M are analytic in μ at $\mu=0$. The transformation (5.1) is well-defined and well-conditioned as $\mu \to 0$, its inverse is readily available in terms of L and M, and its use results in a transformed Kalman filter whose coefficients are analytic in μ which is crucial for obtaining approximations. The use of any transformation outside this class has to be preceded by examining the analytical and numerical properties of the transformation for small μ, which is beyond the goal of our research. Within the class of transformations represented by (5.1) there is arbitrariness concerning the choice of the matrices L and M. Aside from the choice L_1 and M_1 which was adopted in Section III to block diagonalize $(A-BF)$ one might consider several other choices. For example L and M might be chosen to block diagonalize the open-loop state matrix A or the matrix $(A-KC)$ which is the homogeneous part of the Kalman filter if it is viewed as a system driven by both the control input u and the observed output y. They might even be chosen as zeros meaning that the Kalman filter is represented in the original coordinates.[3] With such arbitrariness in choosing L and M we need to get some insight in the effect of using the transformation (5.1) on the decoupling of the state variables of the Kalman filter. If

$$\hat{\eta} \triangleq \begin{pmatrix} \hat{\eta}_1 \\ \hat{\eta}_2 \end{pmatrix} = T\hat{x}$$

is the state vector of the transformed Kalman filter, then the power spectrum of $\hat{\eta}$ can be determined by studying the equation

$$\dot{\hat{\eta}} = T(A-BF)T^{-1}\hat{\eta} + TKv.$$

Since the innovation process v is white noise, the power spectrum of $\hat{\eta}$ is determined by the transfer matrix

$$W(s) = T(sI - A + BF)^{-1}K \triangleq \begin{pmatrix} W_1(s) \\ W_2(s) \end{pmatrix},$$

$$W_1 \in C^{n_1 \times l}, \quad W_2 \in C^{n_2 \times l}.$$

To compute the resolvent matrix $(sI-A+BF)^{-1}$, the transformation (3.26) is used to block diagonalize $(A-BF)$, resolvent matrices of the on-diagonal blocks are computed and then the inverse transformation of (3.26) is applied to recover $(sI-A+BF)^{-1}$. Multiplying $(sI-A+BF)^{-1}$ from the right by K and from the left by T, where T is any member in the class of transformations defined by (5.1), we get

$$\begin{aligned} W_1(s) = (I - \mu ML + ML_1)\Gamma_1(s)(K_1 - \mu M_1 L_1 K_1 - M_1 K_2) \\ + \mu(M_1 - M - \mu MLM_1 + \mu ML_1 M_1) \\ \cdot \Gamma_2(\mu s)(K_2 + \mu L_1 K_1) \end{aligned} \tag{5.2}$$

$$\begin{aligned} W_2(s) = (L - L_1)\Gamma_1(s)(K_1 - \mu M_1 L_1 K_1 - M_1 K_2) \\ + (I + \mu LM_1 - \mu L_1 M_1)\Gamma_2(\mu s)(K_2 + \mu L_1 K_1) \end{aligned} \tag{5.3}$$

where

$$\Gamma_1(s) = [sI - (A_{11} - B_1 F_1) + (A_{12} - B_1 F_2)L_1]^{-1}$$

and

$$\Gamma_2(\mu s) = [\mu sI - (A_{22} - B_2 F_2) - \mu L_1(A_{12} - B_1 F_2)]^{-1}.$$

Examining expressions (5.2) and (5.3) shows that all transformations of the form (5.1) share two properties. First, the fast variable $\hat{\eta}_2$ has a slow bias. Second, the fast component in $W_1(s)$ is multiplied by μ, hence, for sufficiently small μ, $\hat{\eta}_1$ is predominantly slow. This shows that, for sufficiently small μ, implementation of the slow equation of the Kalman filter can be done using an integration step size much larger than that used for integrating the fast equation. Examination of (5.2) and (5.3) shows also that there is one choice of L and M that results in perfect decoupling of the slow and fast variables and that is the choice $L=L_1$ and $M=M_1$ which was adopted in Section III. Of course the final word on the choice of the appropriate coordinates for representing the Kalman filter is problem dependent and implementation factors which are not considered here (e.g., physical constraints, hardware architectures, etc.) will affect the choice. It is clear, however, that from the decoupling viewpoint the transformation used in Section III is the best. The important point we want to make here is that for any other transformation of the form (5.1) the results of Section III are generic in nature, that is, approximation schemes can be defined by truncating expansions of F_1, F_2, K_1, K_2, L, and M exactly as it was done in Section III, and theorems similar to Theorems 3 and 4 can be proved.

VI. Conclusions

A new approach to simplifying LQG optimal control for singularly perturbed systems has been outlined. The new approach alleviates the difficulties of the previous approaches, is conceptually simple, and retains the physically motivated structure of the composite control of [10].

Appendix
Block Diagonalization of Singularly Perturbed Systems

Consider the linear time-invariant singularly perturbed system

$$\dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t) + B_1 u(t) \tag{A1}$$

$$\mu\dot{x}_2(t) = A_{21}x_1(t) + A_{22}x_2(t) + B_2 u(t) \tag{A2}$$

where the matrices A_{ij} and B_i are analytic functions of μ, and $\det[A_{22}(0)] \neq 0$. The state transformation

$$y_2 = x_2 + Lx_1, \tag{A3}$$

where L is chosen to satisfy

$$0 = A_{22}L - A_{21} - \mu L(A_{11} - A_{12}L), \tag{A4}$$

transforms the system (A1), (A2) into

$$\dot{x}_1(t) = (A_{11} - A_{12}L)x_1(t) + A_{12}y_2(t) + B_1 u(t) \tag{A5}$$

$$\mu\dot{y}_2(t) = (A_{22} + \mu LA_{12})y(t) + (B_2 + \mu LB_1)u(t) \tag{A6}$$

which is a lower block triangular matrix. Now, the state transformation

$$y_1 = x_1 - \mu My_2, \tag{A7}$$

where M is chosen to satisfy

[3] In our discussion here the matrices A, B, C, F, and K represent overall system matrices, i.e.,

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21}/\mu & A_{22}/\mu \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2/\mu \end{pmatrix}, \quad C = (C_1 \quad C_2),$$

$$F = (F_1 \quad F_2) \quad \text{and} \quad K = \begin{pmatrix} K_1 \\ K_2/\mu \end{pmatrix}.$$

$$0 = -MA_{22} + A_{12} - \mu MLA_{12} + \mu(A_{11} - A_{12}L)M, \quad \text{(A8)}$$

transforms the system (A5), (A6) into

$$\dot{y}_1(t) = (A_{11} - A_{12}L)y_1(t) + (B_1 - MB_2 - \mu MLB_1)u(t) \quad \text{(A9)}$$

$$\mu \dot{y}_2(t) = (A_{22} + \mu LA_{12})y_2(t) + (B_2 + \mu LB_1)u(t) \quad \text{(A10)}$$

which is a block diagonal matrix. The existence of L and M satisfying (A4) and (A8) is established in two steps. First, at $\mu = 0$, the nonsingularity of $A_{22}(0)$ guarantees that $L(0)$ and $M(0)$ exist and are given by

$$L(0) = [A_{22}(0)]^{-1}A_{21}(0), \quad \text{(A11)}$$

$$M(0) = A_{12}(0)[A_{22}(0)]^{-1}. \quad \text{(A12)}$$

Second, applying the implicit function theorem and using again the nonsingularity of $A_{22}(0)$, it can be shown that there exists $\mu^* > 0$ such that $\forall \mu \in (0, \mu^*)$, there exist $L(\mu)$ and $M(\mu)$ satisfying (A4) and (A8). Moreover, $L(\mu)$ and $M(\mu)$ are analytic functions of μ. The overall transformation is given by

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} I - \mu ML & -\mu M \\ L & I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{(A13)}$$

whose inverse is well-defined and is given by

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} I & \mu M \\ -L & I - \mu LM \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \quad \text{(A14)}$$

References

[1] P. V. Kokotovic, R. O'Malley, and P. Sannuti, "Singular perturbations and order reduction in control theory. An overview," *Automatica*, vol. 12, pp. 123–132, 1976.

[2] P. Sannuti and P. V. Kokotovic, "Near optimum design of linear systems by a singular perturbation method," *IEEE Trans. Automat. Contr.*, vol. AC-14, pp. 15–22, 1969.

[3] R. A. Yackel, "Singular perturbation of the linear state regulator," Ph.D. dissertation, Coordinated Sci. Lab., Univ. Illinois, Urbana, Rep. R-532, Sept. 1971.

[4] P. V. Kokotovic and R. A. Yackel, "Singular perturbation of linear regulators: Basic theorems," *IEEE Trans. Automat. Contr.*, vol. AC-17, pp. 29–37, 1972.

[5] R. A. Yackel and P. V. Kokotovic, "A boundary layer method for the matrix Riccati equation," *IEEE Trans. Automat. Contr.*, vol. AC-18, pp. 17–24, 1973.

[6] R. E. O'Malley, Jr., "Singular perturbations of the time invariant linear state regulator problem," *J. Differential Equations*, vol. 12, pp. 117–128, 1972.

[7] R. R. Wilde and P. V. Kokotovic, "Optimal open- and closed-loop control of singularly-perturbed linear systems," *IEEE Trans. Automat. Contr.*, vol. AC-18, pp. 616–625, 1973.

[8] J. H. Chow and P. V. Kokotovic, "A decomposition of near-optimum regulators for systems with slow and fast modes," *IEEE Trans. Automat. Contr.*, vol. AC-21, Oct. 1976.

[9] A. Haddad, "Linear filtering of singularly perturbed systems," *IEEE Trans. Automat. Contr.*, vol. AC-21, pp. 515–519, 1976.

[10] A. Haddad and P. Kokotovic, "Stochastic control of linear singularly perturbed systems," *IEEE Trans. Automat. Contr.*, vol. AC-22, pp. 815–821, 1977.

[11] D. Teneketzis and N. Sandell, "Linear regulator design for stochastic systems by multiple time-scale method," *IEEE Trans. Automat. Contr.*, vol. AC-22, pp. 615–621, 1977.

[12] K. Chang, "Singular perturbations of a general boundary value problem," *SIAM J. Math. Anal.*, vol. 3, pp. 520–526, 1972.

[13] R. R. Wilde and P. V. Kokotovic, "Stability of singularly perturbed systems and networks with parasitics," *IEEE Trans. Automat. Contr.*, vol. AC-17, pp. 245–246, 1972.

[14] H. Kwakernaak and R. Sivan, *Linear Optimal Control Systems*. New York: Wiley, 1972.

[15] F. Hoppensteadt, "Properties of ordinary differential equations with small parameters," *Comm. Pure Appl. Math.*, vol. XXIV, pp. 807–840, 1971.

[16] H. K. Khalil, *et al.*, "Singular perturbation methods and multimodel control," College of Eng., Michigan State Univ., East Lansing, MI, Res. Rep., Dec. 1983.

[17] P. V. Kokotovic, "A Riccati equation for block-diagonalization of ill-conditioned systems," *IEEE Trans. Automat. Contr.*, vol. AC-20, no. 6, pp. 812–814, 1975.

[18] H. K. Khalil, "Control of linear singularly perturbed systems with colored noise disturbance," *Automatica*, vol. 14, pp. 153–156, 1978.

[19] P. V. Kokotovic and J. B. Cruz, Jr., "An approximation theorem for linear optimal regulators," *J. Math. Anal. Appl.*, vol. 27, pp. 249–252, 1969.

[20] J. O'Reilly, "Dynamical feedback control of singularly perturbed linear systems using a full-order observer," *Int. J. Contr.*, vol. 31, pp. 1–10, 1980.

[21] H. Khalil, A. Haddad, and G. Blankenship, "Parameter scaling and well-posedness of stochastic singularly perturbed control systems," in *Proc. 12th Asilomar Conf.*, Pacific Cove, CA, 1978.

[22] A. Bensoussan, "Singular perturbation results for a class of stochastic control problems," *IEEE Trans. Automat. Contr.*, vol. AC-26, no. 5, pp. 1071–1080, 1981.

[23] V. R. Saksena and J. B. Cruz, Jr., "A multimodel approach to stochastic Nash games," reprints *1981 IEEE Int. Symp. on Circuits Syst.*, Apr. 1981.

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