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REFERENCES

[1] P. Dorato, Ed., *Robust Control*. New York: IEEE Press, 1987.
[2] N. R. Sandell, Jr., Ed., "Recent developments in the robustness theory of multivariable systems," Office of Naval Research, Rep. no. ONR-CR215-271-1F, Aug. 1979.
[3] N. A. Lehtomaki, N. R. Sandell, Jr., and M. Athans, "Robustness results in linear-quadratic-Gaussian-based multivariable control designs," *IEEE Trans. Automat. Contr.*, vol. AC-26, no. 1, pp. 75-93, Feb. 1981.
[4] J. C. Doyle, "Analysis of feedback systems with structured uncertainties," *IEE Proc.*, vol. 129, pt. D, no. 6, pp. 242-250, Nov. 1982.
[5] M. Tahk and J. L. Speyer, "Modeling of parameter variations and asymptotic LQG synthesis," *IEEE Trans. Automat. Contr.*, vol. AC-32, no. 9, pp. 793-801, Sept. 1987.
[6] R. K. Yedavalli and Z. Liang, "Reduced conservatism in stability robustness bounds by state transformation," *IEEE Trans. Automat. Contr.*, vol. AC-31, no. 9, pp. 863-866, Sept. 1986.
[7] I. Horowitz, "Quantitative feedback theory," *IEE Proc.*, vol. 129, pt. D, no. 6, pp. 215-226, Nov. 1982.
[8] E. Yaz, "Deterministic and stochastic robustness measures for discrete systems," *IEEE Trans. Automat. Contr.*, vol. 33, no. 10, pp. 952-955, Oct. 1988.
[9] H. J. Kushner, *Stochastic Stability and Control*. New York: Academic, 1967.
[10] R. F. Stengel, "Some effects of parameter variations on the lateral-directional stability of aircraft," *AIAA J. Guidance Contr.*, vol. 3, no. 2, pp. 124-131, Apr. 1980.
[11] —, *Stochastic Optimal Control: Theory and Application*. New York: Wiley, 1986.
[12] R. F. Stengel and L. E. Ryan, "Stochastic robustness of linear control systems," in *Proc. 1989 Conf. Inform. Sci. Syst.*, Mar. 1989. pp. 556-561.
[13] —, "Multivariate histograms for analysis of linear control system robustness," in *Proc. 1989 Amer. Contr. Conf.*, Pittsburgh, PA, June 1989, pp. 937-943.
[14] —, "Application of stochastic robustness to aircraft control systems," in *Proc. 1989 AIAA Guidance, Navigation, Contr. Conf.*, Boston, MA, Aug. 1989, pp. 698-708.
[15] D. D. Siljak, *Nonlinear Systems, The Parameter Analysis and Design*. New York: Wiley, 1969.
[16] J. Ackermann, "Parameter space design of robust control systems," *IEEE Trans. Automat. Contr.*, vol. AC-25 no. 6, pp. 1058-1072, Dec. 1980.
[17] P. Putz and M. J. Wozny, "A new computer graphics approach to parameter space design of control systems," *IEEE Trans. Automat. Contr.*, vol. AC-32, no. 4, pp. 1058-1072, Apr. 1987.
[18] S. Boyd, V. Balakrishnan, C. Barratt, N. Khraishi, X. Li, D. Meyer, and S. Norman, "A new CAD method and associated architectures for linear controllers," *IEEE Trans. Automat. Contr.*, vol. 33, no. 3, pp. 268-283, Mar. 1988.
[19] W. J. Conover, *Practical Non-Parametric Statistics*. New York: Wiley, 1980.

Parallel Reduced-Order Controllers for Stochastic Linear Singularly Perturbed Discrete Systems

Zoran Gajic and Xuemin Shen

Abstract—**This note presents an approach to the decomposition and approximation of linear quadratic Gaussian control problems for singularly perturbed discrete systems at steady state. The global Kalman filter is decomposed into separate reduced-order local filters via the use of a decoupling transformation. A near-optimal control law is derived by approximating coefficients of the optimal control law. The proposed method allows parallel processing of information and reduces both off-line and on-line computational requirements. A real world example demonstrates the efficiency of the proposed method.**

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I. INTRODUCTION

Linear singularly perturbed discrete systems have been studied in a fast time-scale version [1]-[9] and slow time-scale version (e.g., [10], [11]). Discrete-time models of singularly perturbed linear systems, similar to [10], [11], were studied by Mahmoud and his co-workers [12]. Since the slow time-scale version presupposes the asymptotic stability of the fast modes, it seems, that in the design procedure of stabilizing feedback controllers, the fast time-scale version is much more appropriate [6]. In this note, we will adopt the structure of singularly perturbed discrete linear systems defined by Litkouhi and Khalil [4]-[6], and study corresponding linear-quadratic Gaussian (LQG) control problem.

The continuous-time LQG problem of singularly perturbed systems [14], [15] is solved in [16] by using the power series expansion approach, and later in [17] by using the fixed-point theory. The discrete-time LQG problem of a singularly perturbed system has not been studied, despite the extensive study of the corresponding deterministic counterpart [4]-[6]. We will resolve this problem by using results recently obtained in [18]. The main equation of the optimal linear control theory, the Riccati equation, has a quite complicated form in the discrete-time domain. Partitioning this equation, in the spirit of singular perturbation methodology, will produce a lot of terms (partitioned inversion of a matrix sum) and make the corresponding problem numerically inefficient, even though the problem order reduction is achieved. By applying a bilinear transformation [13], the solution of the discrete algebraic Riccati equation of singularly perturbed systems is obtained in [18] by using already known results for the corresponding continuous-time algebraic Riccati equation [17] (see the Appendix). The method produces the reduced-order near-optimal solution, up to an arbitrary degree of accuracy $O(\epsilon^k)$, where ϵ is a small perturbation parameter and k represents the number of iterations. This reduces the size of required off-line computations and is very suitable for parallel programming.

The importance of the existence of the $O(\epsilon^k)$ theory for singularly perturbed problems is indicated in [19], where the $O(\epsilon)$ theory fails to produce the required result, so that the existence of the $O(\epsilon^k)$ theory is a necessary requirement.

The singularly perturbed structure of the global Kalman filter is exploited in this note, such that it may be replaced by two lower order local filters which will produce additional on-line savings in required computations. This has been achieved via the use of a decoupling transformation introduced in [20], which produces the exact block diagonalization of the global Kalman filter. The approximate feedback control law is obtained by approximating coefficients of the optimal local filters with an accuracy of $O(\epsilon^N)$. The order of approximation of the optimal performance is $O(\epsilon^N)$. The order of approximation of the optimal system trajectories is $O(\epsilon^{N+1/2})$ in the case of slow variables and $O(\epsilon^N)$ in the case of fast variables. All required coefficients of desired accuracy are obtained by using the recursive reduced-order fixed-point type numerical techniques [17], [18], and [21]. Obtained numerical algorithms converge to required optimal coefficients with the rate of convergence of $O(\epsilon)$.

A real world example, a fifth-order discrete model of a steam power system [23], is included in the note, in order to demonstrate the efficiency of the proposed method.

II. LINEAR QUADRATIC GAUSSIAN CONTROL OF DISCRETE SINGULARLY PERTURBED SYSTEMS AT THE STEADY STATE

Consider the singularly perturbed discrete linear stochastic system represented in the fast time-scale by [4]-[6], [18], [25]

$$x_1(n+1) = (I + \epsilon A_{11})x_1(n) + \epsilon A_{12}x_2(n) + \epsilon B_1 u(n) + \epsilon G_1 w(n) \quad (1a)$$

$$x_2(n+1) = A_{21}x_1(n) + A_{22}x_2(n) + B_2 u(n) + G_2 w(n) \quad (1b)$$

$$y(n) = C_1 x_1(n) + C_2 x_2(n) + v(n) \quad (2)$$

with the performance criterion

$$J = E\left\{ \sum_{n=0}^{\infty} \left[z^T(n) z(n) + u^T(n) R u(n) \right] \right\}, \qquad R > 0 \quad (3)$$

where $x_i \in \boldsymbol{R}^{n_i}$, $i = 1, 2$, comprise slow and fast state vectors, respectively, $u \in \boldsymbol{R}^m$ is the control input, $y \in \boldsymbol{R}^l$ is the observed output, $w \in \boldsymbol{R}^r$ and $v \in \boldsymbol{R}^l$ are independent zero-mean stationary white Gaussian noise mutually uncorrelated processes with intensities $W > 0$ and $V > 0$, respectively, and $z \in \boldsymbol{R}^s$ is the controlled output given by

$$z(n) = D_1 x_1(n) + D_2 x_2(n). \quad (4)$$

All matrices are bounded functions of a small positive parameter ϵ [17] having appropriate dimensions.

The optimal control law is given by [22]

$$u(n) = -F\hat{x}(n) \quad (5)$$

with

$$\hat{x}(n+1) = A\hat{x}(n) + Bu(n) + K[y(n) - C\hat{x}(n)] \quad (6)$$

where

$$A = \begin{bmatrix} I + \epsilon A_{11} & \epsilon A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} \epsilon B_1 \\ B_2 \end{bmatrix}, \quad C = [C_1 \quad C_2]$$

$$K = \begin{bmatrix} \epsilon K_1 \\ K_2 \end{bmatrix}, \quad F = [F_1 \quad F_2].$$

Regulator gain F and filter gain K are obtained from

$$F = (R + B^T P B)^{-1} B^T P A \quad (7)$$

$$K = AQC^T(V + CQC^T)^{-1} \quad (8)$$

where P and Q are positive semidefinite stabilizing solutions of the discrete-time algebraic regulator and filter Riccati equations, respectively, given by

$$P = D^T D + A^T P A - A^T P B (R + B^T P B)^{-1} B^T P A \quad (9)$$

$$Q = AQA^T - AQC^T(V + CQC^T)^{-1} CQA^T + GWG^T \quad (10)$$

where

$$D = [D_1 \quad D_2], \quad G = \begin{bmatrix} \epsilon G_1 \\ G_2 \end{bmatrix}.$$

Due to the singularly perturbed structure of the problem matrices, the required solutions P and Q in the fast time-scale version have the form

$$P = \begin{bmatrix} P_{11}/\epsilon & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}, \quad Q = \begin{bmatrix} \epsilon Q_{11} & \epsilon Q_{12} \\ \epsilon Q_{12}^T & Q_{22} \end{bmatrix}. \quad (11)$$

In order to obtain required solutions of (9), (10) in terms of the reduced-order problems and overcome the complicated partitioned form of the discrete-time algebraic Riccati equation, we have used the method developed in [18] (based on a bilinear transformation [13]), to transform the discrete algebraic Riccati equations (9), (10) into continuous-time algebraic Riccati equations (see the Appendix).

Getting approximate solutions for P and Q in terms of reduced-order problems will produce savings in off-line computations. However, in the case of stochastic systems, where an additional dynamical system—filter—has to be built, one is particularly interested in the reduction of on-line computations. This will be achieved by using a decoupling transformation introduced in [20]. The Kalman filter (6) is viewed as a system driven by the innovation process [16]. However, one might study the filter form when it is driven by both measurement and control. The filter form under consideration is obtained from (6) as

$$\hat{x}_1(n+1) = (I + \epsilon A_{11} - \epsilon B_1 F_1)\hat{x}_1(n) + \epsilon(A_{12} - B_1 F_2)\hat{x}_2(n) + \epsilon K_1 \nu(n) \quad (12.a)$$

$$\hat{x}_2(n+1) = (A_{21} - B_2 F_1)\hat{x}_1(n) + (A_{22} - B_2 F_2)\hat{x}_2(n) + K_2 \nu(n) \quad (12.b)$$

with the innovation process

$$\nu(n) = y(n) - C_1\hat{x}_1(n) - C_2\hat{x}_2(n). \quad (13)$$

The nonsingular state transformation of [20] will block diagonalize (12). That transformation is given by

$$\begin{bmatrix} \hat{\eta}_1(n) \\ \hat{\eta}_2(n) \end{bmatrix} = \begin{bmatrix} I_1 - \epsilon HL & -\epsilon H \\ L & I_2 \end{bmatrix} \begin{bmatrix} \hat{x}_1(n) \\ \hat{x}_2(n) \end{bmatrix} = T \begin{bmatrix} \hat{x}_1(n) \\ \hat{x}_2(n) \end{bmatrix} \quad (14)$$

with

$$T^{-1} = \begin{bmatrix} I_1 & \epsilon H \\ -L & I_2 - \epsilon LH \end{bmatrix}$$

where matrices L and H satisfy equations

$$\epsilon L a_{11} + (I - a_{22})L + a_{21} - \epsilon L a_{12} L = 0 \quad (15)$$

$$H(I - a_{22} - \epsilon L a_{12}) + \epsilon(a_{11} - a_{12}L)H + a_{12} = 0 \quad (16)$$

with

$$a_{11} = A_{11} - B_1 F_1, \qquad a_{12} = A_{12} - B_1 F_2$$

$$a_{21} = A_{21} - B_2 F_1, \qquad a_{22} = A_{22} - B_2 F_2.$$

The optimal feedback control, expressed in the new coordinates, has the form

$$u(n) = -f_1\hat{\eta}_1(n) - f_2\hat{\eta}_2(n) \quad (17)$$

with

$$\hat{\eta}_1(n+1) = \alpha_1\hat{\eta}_1(n) + \epsilon\beta_1\nu(n) \quad (18.a)$$

$$\hat{\eta}_2(n+1) = \alpha_2\hat{\eta}_2(n) + \beta_2\nu(n) \quad (18.b)$$

where

$$f_1 = F_1 - F_2 L, \quad f_2 = F_2 + \epsilon(F_1 - F_2 L)H$$

$$\alpha_1 = I + \epsilon(a_{11} - a_{12}L), \quad \alpha_2 = a_{22} + \epsilon L a_{12}$$

$$\beta_1 = K_1 - H(K_2 + \epsilon L K_1), \quad \beta_2 = K_2 + \epsilon L K_1.$$

The innovation process ν is now given by

$$\nu(n) = y(n) - d_1\hat{\eta}_1(n) - d_2\hat{\eta}_2(n) \quad (19)$$

where

$$d_1 = C_{11} - \epsilon C_2 L, \quad d_2 = C_2 + \epsilon(C_1 - C_2 L)H.$$

Approximate control law is defined by perturbing coefficients F_i, $K_i (i = 1, 2)$, L, and H with $O(\epsilon^k)$, $k = 1, 2, \cdots$, in other words, by using kth approximations for these coefficients, where k stands for the required order of accuracy

$$u^{(k)}(n) = -f_1^{(k)}\hat{\eta}_1^{(k)}(n) - f_2^{(k)}\hat{\eta}_2^{(k)}(n) \quad (20)$$

with

$$\hat{\eta}_1^{(k)}(n+1) = \alpha_1^{(k)}\hat{\eta}_1^{(k)}(n) + \epsilon\beta_1^{(k)}\nu^{(k)}(n) \quad (21.a)$$

$$\hat{\eta}_2^{(k)}(n+1) = \alpha_2^{(k)}\hat{\eta}_2^{(k)}(n) + \beta_2^{(k)}\nu^{(k)}(n) \quad (21.b)$$

where

$$\nu^{(k)}(n) = y(n) - d_1^{(k)}\hat{\eta}_1^{(k)}(n) - d_2^{(k)}\hat{\eta}_2^{(k)}(n) \quad (22)$$

and

$$f_i^{(k)} = f_i + O(\epsilon^k), \quad d_i^{(k)} = d_i + O(\epsilon^k)$$

$$\beta_i^{(k)} = \beta_i + O(\epsilon^k), \quad \alpha_i^{(k)} = \alpha_i + O(\epsilon^k) \qquad i = 1, 2.$$

The cost under the approximate control is obtained by using standard routines [22].

The near optimality of the proposed approximate control law (20) is established in the following theorem [18].

Theorem 1: Let x_1 and x_2 be the optimal trajectories and J be the optimal value of the performance criterion. Let $x_1^{(k)}$, $x_2^{(k)}$, and $J^{(K)}$ be the corresponding quantities under the approximate control law $u^{(k)}$ given by (20). Under the conditions stated in Assumption 1 (see the Appendix) the following holds:

$$J^{\text{opt}} - J^{(k)} = O(\epsilon^k) \quad (23.a)$$

$$\text{var}\{x_1 - x_1^{(k)}\} = O(\epsilon^{2k+1}) \quad (23.b)$$

$$\text{var}\{x_2 - x_2^{(k)}\} = O(\epsilon^{2k}) \qquad k = 0, 1, 2, \cdots. \quad (23.c)$$

The proof of this theorem is rather lengthy and is omitted. However, the proof follows the ideas of Theorems 1 and 2 [16]. In addition, due to the discrete nature of the problem, the proof of our Theorem 1 utilizes the bilinear transformation [24] which transforms the discrete Lyapunov equation into the continuous one and compares it to the corresponding equation under the optimal control law. This has been discussed in [18].

III. Numerical Example

A real world physical example (a fifth-order discrete model of a steam power system [23]) demonstrates the efficiency of the proposed method. The problem matrices A and B are given by

$$A = \begin{bmatrix} 0.9150 & 0.0510 & 0.0380 & 0.0150 & 0.0380 \\ -0.0300 & 0.8890 & -0.0005 & 0.0460 & 0.1110 \\ -0.0060 & 0.4680 & 0.2470 & 0.0140 & 0.0480 \\ -0.7150 & -0.0220 & -0.0211 & 0.2400 & -0.0240 \\ -0.1480 & -0.0030 & -0.0040 & 0.0900 & 0.0260 \end{bmatrix}$$

$$B^T = [0.0098 \;\; 0.1220 \;\; 0.0360 \;\; 0.5620 \;\; 0.1150].$$

Remaining matrices are chosen as

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \quad D^TD = \text{diag}\{5, 5, 5, 5, 5\} \quad R = I.$$

It is assumed that $G = B$, and that white noise intensity matrices are given by

$$W = 5.0, \quad V_1 = 5.0, \quad V_2 = 5.0.$$

It is shown in [23] that this model has the singularly perturbed form with $n_1 = 2$, $n_2 = 3$, and $\epsilon = 0.264$. Simulation results are presented in Table I.

It can be seen from Table I that we have quite rapid convergence to the optimal solution. This table justifies the result of Theorem 1, that $J^{(k)} - J^{\text{opt}} = O(\epsilon^k)$. Note that $(0.264)^6 = 3 \times 10^{-4}$.

TABLE I
Approximate Values for Criterion

k	$J^{(k)}$	$J^{(k)} - J^{\text{opt}}$
0	13.4918	0.229×10^{-1}
1	13.4825	0.136×10^{-1}
2	13.4700	0.110×10^{-2}
3	13.4695	0.600×10^{-3}
4	13.4690	1.000×10^{-4}
5	13.4689	$< 10^{-4}$
optimal	13.4689	

IV. Conclusion

The near-optimum (up to any desired accuracy) steady-state regulators are obtained for stochastic linear singularly perturbed discrete systems. The proposed method considerably reduces the size of required off-line and on-line computations by introducing full parallelism in the design procedure.

Appendix

Under a bilinear transformation defined in [13], the algebraic discrete Riccati equation (9) is transformed into the continuous one

$$A_c^T P_c + P_c A_c + Q_c - P_c S_c P_c = 0, \quad S_c = B_c R_c^{-1} B_c^T \quad (A.1)$$

such that $P = P_c$. It has been shown that (A.1) preserves the structure of singularly perturbed systems [18], namely

$$A_c = \begin{bmatrix} O(\epsilon) & O(\epsilon) \\ O(1) & O(1) \end{bmatrix}, \quad B_c = \begin{bmatrix} O(\epsilon) \\ O(1) \end{bmatrix}, \quad Q_c = \begin{bmatrix} O(1) & O(1) \\ O(1) & O(1) \end{bmatrix} \quad (A.2)$$

so that the reduced-order recursive algorithm developed in [17] can be used for solving (A.1). The solution converges with the rate of convergence of $O(\epsilon)$ under the following assumption [18].

Assumption 1: The slow and fast subsystems are stabilizable-detectable and the fast subsystem matrix has no eigenvalues located at -1.

By the duality property between the filter and regulator Riccati equations, the same algorithm can be used for the solution of (10).

More about the use of a bilinear transformation in a singularly perturbed linear-quadratic control problem can be found in [25].

References

[1] V. Butuzov and A. Vasileva, "Differential and difference equation systems with a small parameter for the case in which the unperturbed (singular) system is in the spectrum," *J. Differential Equations*, vol. 6, pp. 499–510, 1971.

[2] F. Hoppensteadt and W. Miranker, "Multitime methods for systems of difference equations," *Studies Appl. Math.*, vol. 56, pp. 273–289, 1977.

[3] G. Blankenship, "Singularly perturbed difference equations in optimal control problems," *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 911–917, 1981.

[4] B. Litkouhi, "Sampled-data control of systems with slow and fast modes," Ph.D. dissertation, Michigan State Univ., East Lansing, MI, 1983.

[5] B. Litkouhi and H. Khalil, "Infinite-time regulators for singularly perturbed difference equations," *Int. J. Contr.*, vol. 39, pp. 587–598, 1984.

[6] ——, "Multirate and composite control of two-time-scale discrete systems," *IEEE Trans. Automat. Contr.*, vol. AC-30, pp. 645–651, 1985.

[7] M. Mahmoud, "Stabilization of discrete systems with multiple-time scales," *IEEE Trans. Automat. Contr.*, vol. AC-31, pp. 159–162, 1986.

[8] H. Oloomi and M. Sawan, "The observer-based controller design of discrete-time singularly perturbed systems," *IEEE Trans. Automat. Contr.*, vol. AC-32, pp. 246–248, 1987.

[9] K. Khorasani and M. Azimi-Sadjadi, "Feedback control of two-time-scale block implemented discrete-time systems," *IEEE Trans. Automat. Contr.*, vol. AC-32, pp. 69–73, 1987.

[10] R. Phillips, "Reduced-order modeling and control of two time scale discrete control systems," *Int. J. Contr.*, vol. 31, pp. 761–780, 1980.

[11] D. Naidu and Rao, "Singular perturbation analysis of discrete control sytems," in *Lecture Notes in Mathematics, Vol. 1154*. New York, Springer-Verlag, 1985.

[12] M. Mahmoud, Y. Chen, and M. Singh, "Discrete two-time-scale systems," *Int. J. Syst. Sci.*, vol. 17, pp. 1187–1207, 1986.

[13] R. Kondo and K. Furuta, "On the bilinear transformation of Riccati equations," *IEEE Trans. Automat. Contr.*, vol. AC-31, pp. 50–54, 1986.

[14] P. Kokotovic and H. Khalil, *Singular Perturbations in Systems and Control*. New York: IEEE Press, 1986.
[15] P. Kokotovic, H. Khalil, and J. O'Reilly, *Singular Perturbation Methods in Control: Analysis and Design*. New York: Academic, 1986.
[16] H. Khalil and Z. Gajic, "Near-optimum regulators for stochastic linear singularly perturbed systems," *IEEE Trans. Automat. Contr.*, vol. AC-29, pp. 531-541, 1984.
[17] Z. Gajic, "Numerical fixed-point solution for near-optimum regulators of linear quadratic Gaussian control problems for singularly perturbed systems," *Int. J. Contr.*, vol. 43, pp. 373-387, 1986.
[18] X. Shen, "Near-optimum reduced-order stochastic control of linear discrete and continuous systems with small parameters," Ph.D. dissertation, Rutgers Univ., Piscataway, NJ, 1990.
[19] Z. Gajic, Dj. Petkovski, and N. Harkara, "The recursive algorithm for the optimal static output feedback control problem of linear singularly perturbed systems," *IEEE Trans. Automat. Contr.*, vol. 34, pp. 465-468, Apr. 1989.
[20] K. Chang, "Singular perturbations of a general boundary value problem," *SIAM J. Math. Anal.*, vol. 3, pp. 520-526, 1972.
[21] T. Grodt and Z. Gajic, "The recursive reduced-order numerical solution of the singularly perturbed matrix differential Riccati equation," *IEEE Trans. Automat. Contr.*, vol. 33, pp. 751-754, 1988.
[22] H. Kwakernaak and R. Sivan, *Linear Optimal Control Systems*. New York: Wiley, 1972.
[23] M. Mahmoud, "Order reduction and control of discrete systems," *Proc. IEE*, vol. 129, part D, pp. 129-135, 1982.
[24] H. M. Power, "Equivalence of Lyapunov matrix equations for continuous and discrete systems," *Electron. Lett.*, vol. 3, p. 83, 1967.
[25] Z. Gajic, Dj. Petkovski, and X. Shen, *Singularly Perturbed and Weakly Coupled Linear Control System: A Recursive Approach*. New York: Springer-Verlag, 1990.

Noninteracting Control of 2-D Systems

Ettore Fornasini and Giovanni Marchesini

Abstract—**Necessary and sufficient conditions for the existence of a decoupling bicausal precompensator for multivariable 2-D systems are derived in state space and frequency domains. In general, the decoupling problem for 2-D systems can be solved by feedback compensators if suitable injectivity assumptions are introduced on the input-state matrices. The structure of dynamic compensators is derived for this case and the 2-D decoupling problem with stability is solved.**

I. Introduction

Since the late sixties, the decoupling problem constitutes one of the most attractive research topics in multivariable 1-D systems theory. Besides several appealing consequences in the applications, the interest in this field relies on the analytical tools that have been introduced in developing the underlying theory. The decoupling schemes considered in the literature have different characteristics. These include the topology of the interconnections (based on the use of precompensators, feedback compensators, or compound strategies), the dynamical characteristics of the subsystems that enter in the interconnections, the use of state-space or input-output models and, finally, the algebraic structures (fields, rings) which provide the framework where the systems are defined [1]-[5]. In most applications, we are required to solve at the same time the decoupling and the stabilization problems. In these cases, state or output feedbacks have to be considered and only those schemes that include dynamic compensators become relevant to the solution.

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2-D systems provide input-output and state-space models representing physical processes which depend on two independent variables. In some cases, one of these variables is time and the other represents a spatial dimension (as in the study of some classes of distributed parameter systems and delay differential systems), while for other problems, such as image processing, none of the independent variables can be sought of as time. Typically, they apply to two-dimensional data processing in several fields, such as seismology, X-ray image enhancement, image deblurring, digital picture processing, etc. Also, 2-D systems constitute a natural framework for modeling multivariable networks, large scale systems obtained by interconnecting many subsytems and, in general, physical processes where both space and time have to be taken into account [6], [7].

Recently, the feedback control theory of 2-D systems attracted the interest of research people and a great deal of attention has been devoted to problems related to stabilization and characterization of closed-loop characteristic polynomials [8]-[11]. Moreover, the systematic application of 2-D polynomial matrices techniques allowed us to extend the original single-input single-output analysis to include multivariable 2-D systems.

In this note, we aim to analyze how 2-D compensators apply to noninteracting control of multivariable 2-D systems and to find necessary and sufficient conditions for the existence of a feedback law that makes the closed-loop transfer matrix diagonal and nonsingular. We shall tackle this problem using MFD's in two variables, applied to input-output and state-space models. It is worthwhile to remark that several equivalent strategies, based on bicausal precompensators, static precompensators and compensators, static precompensators and dynamic compensators can be implemented in generating noninteracting controls for 1-D systems. As we shall see, in the case of 2-D systems these strategies are not equivalent, since they allow decoupling of different classes of systems.

The state equation of a multivariable 2-D system $\Sigma = (A_1, A_2, B_1, B_2, C, D)$ having m inputs and m outputs is given by

$$\begin{aligned} x(h+1, k+1) &= A_1 x(h+1, k) + A_2 x(h, k+1) \\ &\quad + B_1 u(h+1, k) + B_2 u(h, k+1) \\ y(h, k) &= Cx(h, k) + Du(h, k) \end{aligned} \tag{1.1}$$

where u and y are the m-dimensional vectors of input and output values, x is an n-dimensional local state vector, and A_1, A_2, B_1, B_2, C, D are matrices of appropriate dimensions. In the following, we shall adopt the standard convention that a scalar sequence $\{s(h, k)\}$ with nonnegative indexes h, k is associated with a formal power series $\Sigma s(h, k) z_1^h z_2^k$ having nonnegative powers in z_1 and z_2. According to this convention, a proper (strictly proper) rational function can be represented as a quotient $p(z_1, z_2)/q(z_1, z_2)$ of coprime polynomials with $q(0, 0) \neq 0$ ($q(0, 0) \neq 0$ and $p(0, 0) = 0$).

Therefore, the transfer matrix of Σ is the $m \times m$ rational matrix

$$W(z_1, z_2) = C(I - A_1 z_1 - A_2 z_2)^{-1}(B_1 z_1 + B_2 z_2) + D \tag{1.2}$$

whose entries are proper rational functions in two variables. The system (1.1) is called *strictly proper* if $D = 0$ and *bicausal* if D is an invertible matrix. It is immediate to see that Σ is strictly proper if $W(0, 0) = 0$ and bicausal if $W(0, 0)$ is an invertible matrix.

Because of the structure of 2-D systems, a number of different state feedback schemes is allowed. The simplest of these is represented by the static control law

$$u(h, k) = Kx(h, k), \qquad K \in \mathbf{R}^{m \times n}. \tag{1.3}$$

Comparing to static state feedback in 1-D theory, the possibilities of modifying the dynamical behavior by applying (1.3) are much poorer [12].

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