example, the point of the intersection of line ($\alpha = 4.0$, $r = 11\pi$) with surface M corresponds to $d \approx 1.1\pi$ and, hence, $\epsilon \approx 0.1$. Thus, system (4) with A and B given by (15), $r = 11\pi$, and $D(t/\epsilon)$ given by (13) with $\alpha = 4.0$ and $\tau = t/\epsilon$ is asymptotically stable for any positive $\epsilon \leq 0.1$.

V. Conclusions

This note presents a constructive tool for the stability analysis of periodic linear time lag systems of the form (4). It shows that stability properties of this class of systems are sensitive to small delays, and therefore caution should be exercised in applying vibrational and fast periodic feedback controllers designed under the no delay assumption via results of [2]-[4], to systems with small delays.

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The Recursive Algorithm for the Optimal Static Output Feedback Control Problem of Linear Singularly Perturbed Systems

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Abstract—**The recursive algorithm is developed for solving the algebraic equations comprising the solution of the optimal static output feedback control problem of singularly perturbed linear systems. The proposed algorithm is very efficient from the numerical point of view, since only low-order systems are involved in algebraic calculations and the required solution can be easily obtained up to an arbitrary order of accuracy, that is, $O(\epsilon^k)$ where ϵ is a small perturbation parameter. The real world example demonstrates the failure of $O(\epsilon)$ theory—used so far in the study of this problem, and the necessity for the existence of $O(\epsilon^k)$ theory.**

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I. Introduction

In the early 1970's, increasing attention was given to the problem of designing output constrained regulators where a very limited number of state measurements are available for control implementation (e.g., [1]-[4]). The optimal solution for this control problem is obtained in terms of high-order nonlinear matrix algebraic equations. The convergence complexities of the algorithms proposed for the solution of these equations have hindered for quite a long time a wider application of this technique. Recently, the convergence problem was solved in [5].

The output feedback control problem attracted the attention of the researchers from the field of singular perturbations in the 1980's [6]-[10], [14]. It is well known that the singularly perturbed systems belong to the class of systems with ill-conditioned dynamics which makes corresponding numerical problems stiff. Thus, in addition to the high-order nonlinear matrix algebraic equations, one is faced with the ill-defined numerical problems also.

Motivated by the results of [11]-[13] and [5], we have developed the well-defined recursive numerical technique for the solution of nonlinear algebraic matrix equations associated with the output feedback control problem of linear-quadratic singularly perturbed systems. Moreover, the numerical slow-fast decomposition has been achieved so that only low-order systems are involved in algebraic computations. It is shown that each iteration step of the proposed algorithm improves the accuracy by an order of magnitude, that is, the accuracy of $O(\epsilon^k)$ where ϵ is a small perturbation parameter, can be obtained by performing only k iterations. This represents a significant improvement since all results on the output feedback control problems for the singularly perturbed systems have been obtained so far with an accuracy of $O(\epsilon)$ only.

The real world example, an industrially important reactor, which demonstrates the efficiency of the proposed algorithm and the failure of $O(\epsilon)$ theory is included in the note.

II. Output Feedback Control for Singularly Perturbed Linear Systems

Consider the singularly perturbed linear system [15]

$$\dot{x}_1 = A_1x_1 + A_2x_2 + B_1u, \qquad x_1(t_0) = x_{10} \tag{1}$$

$$\epsilon\dot{x}_2 = A_3x_1 + A_4x_2 + B_2u, \qquad x_2(t_0) = x_{20} \tag{2}$$

$$y = C_1x_1 + C_2x_2 \tag{3}$$

where $x_1 \in R^{n_1}$ and $x_2 \in R^{n_2}$ are state vectors, $u \in R^m$ is a control input, and $y \in R^r$ is a measured output. In the following, A_i, B_j, and C_j, $i = 1, \cdots, 4$, $j = 1, 2$ are constant matrices of compatible dimensions; in general, they are continuous functions of a small positive parameter ϵ [11]. With (1)-(3), consider the performance criterion

$$J = \int_0^\infty \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T Q \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u^TRu \right\} dt \tag{4}$$

with positive definite R and positive semidefinite Q, which has to be minimized. In addition, the control input $u(t)$ is constrained to

$$u(t) = Fy(t). \tag{5}$$

The optimal constant output feedback gain F is given by [1]

$$F = R^{-1}B^TKLC^T(CLC^T)^{-1} \tag{6}$$

where matrices K and L satisfy high-order nonlinear coupled algebraic equations

$$(A - BFC)L + L(A - BFC)^T + x_0x_0^T = 0 \tag{7}$$

$$(A - BFC)^TK + K(A - BFC) + Q + C^TF^TRFC = 0 \tag{8}$$

and newly defined matrices as

$$A=\begin{bmatrix} A_1 & A_2 \\ \frac{A_3}{\epsilon} & \frac{A_4}{\epsilon}\end{bmatrix},\quad B=\begin{bmatrix} B_1 \\ \frac{B_2}{\epsilon}\end{bmatrix},\quad C=[C_1\ \ C_2],\ x_0=\begin{bmatrix} x_{10}\\ x_{20}\end{bmatrix}. \tag{9}$$

Compatible with the nature of their solution, matrices K and L are partitioned as follows:

$$K=\begin{bmatrix} K_1 & \epsilon K_2 \\ \epsilon K_2^T & \epsilon K_3\end{bmatrix},\quad L=\begin{bmatrix} L_1 & L_2 \\ L_2^T & L_3\end{bmatrix}. \tag{10}$$

In a recent paper [5], it is shown that the algorithm proposed for the numerical solution of (6)–(8), defined by

$$\text{choose } F^{(0)} \text{ such that } A-BF^{(0)}C \text{ is a stable matrix} \tag{11}$$

$$(A-BF^{(i)}C)L^{(i+1)}+L^{(i+1)}(A-BF^{(i)}C)^T+x_0x_0^T=0 \tag{12}$$

$$(A-BF^{(i)}C)^TK^{(i+1)}+K^{(i+1)}(A-BF^{(i)}C)+Q+C^TF^{(i)T}RF^{(i)}C=0 \tag{13}$$

$$F^{(i+1)}=R^{-1}B^TK^{(i+1)}L^{(i+1)}C^T(CL^{(i+1)}C^T)^{-1} \tag{14}$$

with $i=1, 2, \cdots$, converges to a local minimum under the nonrestrictive assumption. As a matter of fact, the updated value for F is defined in [5] as

$$F_{\text{v}}^{(i+1)}=F^{(i)}+\alpha(F^{(i+1)}-F^{(i)}) \tag{15}$$

where $\alpha \in (0, 1]$ is chosen at each iteration to ensure that the minimum is not overshot, that is,

$$J_{i+1}=\text{tr}\,\{K^{(i+1)}x_0x_0^T\}<J_i=\text{tr}\,\{K^{(i)}x_0x_0^T\}. \tag{16}$$

It has been customary in the control literature on the output feedback to assume that the initial conditions are uniformly distributed on the unit sphere, that is,

$$x_0x_0^T=I_{(n_1+n_2)}. \tag{17}$$

Applying the slow-fast decomposition transform of Chang [15] to problem (1)–(5) and finding the optimal gains for the slow and fast subsystem is possible for the accuracy of $O(\epsilon)$ only [8]–[9]. It represents a well-posed problem, but there is no way to improve the approximation to any desired order of accuracy, that is, $O(\epsilon^k)$. In this note, we will achieve that goal through the numerical slow–fast decomposition of the algebraic equations (11)–(15).

In order to simplify derivations, we introduce the notation

$$A-BFC=\begin{bmatrix} D_1 & D_2 \\ \frac{D_3}{\epsilon} & \frac{D_4}{\epsilon}\end{bmatrix}=\begin{bmatrix} A_1-B_1FC_1 & A_2-B_1FC_2 \\ \frac{A_3-B_2FC_1}{\epsilon} & \frac{A_4-B_2FC_2}{\epsilon}\end{bmatrix} \tag{18}$$

$$Q+C^TF^TRFC=\begin{bmatrix} q_1 & q_2 \\ q_3 & q_4\end{bmatrix}=\begin{bmatrix} Q_1+C_1^TF^TRFC_1 & Q_2+C_1^TF^TRFC_2 \\ Q_2^T+C_2^TF^TRFC_1 & Q_3+C_2^TF^TRFC_2\end{bmatrix} \tag{19}$$

with obvious definitions for D_i's and q_i's, $i=1, 2, 3, 4$.

Partitioning (12)–(13) compatible to (9)–(10) and using (17)–(19) will produce the following set of equations:

$$D_1^{(i)}L_1^{(i+1)}+L_1^{(i+1)}D_1^{(i)T}+D_2^{(i)}L_2^{(i+1)T}+L_2^{(i+1)}D_2^{(i)T}+I=0 \tag{20a}$$

$$L_2^{(i+1)}D_4^{(i)T}+\epsilon D_1^{(i)}L_2^{(i+1)}+L_1^{(i+1)}D_3^{(i)T}+\epsilon D_2^{(i)}L_3^{(i+1)}=0 \tag{20b}$$

$$L_3^{(i+1)}D_4^{(i)T}+D_4^{(i)}L_3^{(i+1)}+D_3^{(i)}L_2^{(i+1)}+L_2^{(i+1)T}D_3^{(i)T}+\epsilon I=0 \tag{20c}$$

and

$$D_1^{(i)T}K_1^{(i+1)}+K_1^{(i+1)}D_1^{(i)}+D_3^{(i)T}K_2^{(i+1)T}+K_2^{(i+1)}D_3^{(i)}+q_1^{(i)}=0 \tag{21a}$$

$$K_2^{(i+1)}D_4^{(i)}+\epsilon D_1^{(i)T}K_2^{(i+1)}+D_3^{(i)T}K_3^{(i+1)}+K_1^{(i+1)}D_2^{(i)}+q_2^{(i)}=0 \tag{21b}$$

$$K_3^{(i+1)}D_4^{(i)}+D_4^{(i)T}K_3^{(i+1)}+\epsilon D_2^{(i)T}K_2^{(i+1)}+\epsilon K_2^{(i+1)T}D_2^{(i)}+q_3^{(i)}=0 \tag{21c}$$

where

$$D_1^{(i)}=A_1-B_1F^{(i)}C_1,\quad D_2^{(i)}=A_2-B_1F^{(i)}C_2$$

$$D_3^{(i)}=A_3-B_2F^{(i)}C_1,\quad D_4^{(i)}=A_4-B_2F^{(i)}C_2$$

and

$$q_1^{(i)}=Q_1+C_1^TF^{(i)T}RF^{(i)}C_1$$

$$q_2^{(i)}=Q_2+C_1^TF^{(i)T}RF^{(i)}C_2$$

$$q_3^{(i)}=Q_3+C_2^TF^{(i)T}RF^{(i)}C_2,\qquad i=1, 2, 3, \cdots.$$

Since the matrix $A-BF^{(i)}C$ has n_1 slow eigenvalues of $O(1)$ and n_2 fast eigenvalues of $O(1/\epsilon)$, then det $(A-BF^{(i)}C)$ is of $O(1/\epsilon^{n_2})$, which makes (12) and (13) numerically ill defined. However, the partitioned forms of (12) and (13) given by (20) and (21), obtained after multiplying equations for L_2 (K_2) and L_3 (K_3) by ϵ, comprise the well-defined numerical problems, but there are no available methods for their solution. In the next section, we will derive the efficient numerical scheme for solving (20) and (21). Even more, the slow-fast decomposition will be achieved, and the required solutions will be obtained in terms of low-order problems of dimensions n_1 and n_2—the original problems (20) and (21) are of dimensions n_1+n_2.

III. The Recursive Algorithm for the Optimal Output Feedback Control Problem of Singularly Perturbed Linear Systems

Equation (21) is a standard Lyapunov equation of singularly perturbed linear systems. It is a special case of the more general Lyapunov equation studied in [12] and [13]. Its zero-order solution is obtained by setting $\epsilon=0$ in (21), which after some algebra produces

$$\bar{K}_1^{(i+1)}D_0^{(i)}+D_0^{(i)T}\bar{K}_1^{(i+1)}+G_0^{(i)T}G_0^{(i)}=0 \tag{22a}$$

$$\bar{K}_3^{(i+1)}D_4^{(i)}+D_4^{(i)T}\bar{K}_3^{(i+1)}+q_3^{(i)}=0 \tag{22c}$$

$$\bar{K}_2^{(i+1)}=-(\bar{K}_1^{(i+1)}D_2^{(i)}+D_3^{(i)T}\bar{K}_3^{(i+1)}+q_2^{(i)})D_4^{(i)^{-1}} \tag{22b}$$

where

$$D_0^{(i)}=D_1^{(i)}-D_2^{(i)}D_4^{(i)^{-1}}D_3^{(i)}$$

$$G_0^{(i)}=G_1^{(i)}-G_3^{(i)}D_4^{(i)^{-1}}D_3^{(i)},\qquad G_p^{(i)}=\sqrt{q_p^{(i)}},\ p=1, 3.$$

Note that there is no need to calculate the square root of $q_p^{(i)}$'s. The expression for $G_0^{(i)}$ is used in (22a) only to simplify notation, but not for real calculations since $q_1^{(i)}=G_1^{(i)T}G_1^{(i)}$, $q_2^{(i)}=G_1^{(i)T}G_3^{(i)}$, and $q_3^{(i)}=G_3^{(i)T}G_3^{(i)}$. The zero-order solution

$$\bar{K}^{(i+1)}=\begin{bmatrix} \bar{K}_1^{(i+1)} & \epsilon\bar{K}_2^{(i+1)} \\ \epsilon\bar{K}_2^{(i+1)T} & \epsilon\bar{K}_3^{(i+1)}\end{bmatrix} \tag{23}$$

is $O(\epsilon)$ close to the required one $K^{(i+1)}$. We can relate them through the error term E:

$$\epsilon E=K^{(i+1)}-\bar{K}^{(i+1)} \tag{24}$$

or by using a compatible partition:

$$\begin{bmatrix} \epsilon E_1 & \epsilon^2 E_2 \\ \epsilon^2 E_2^T & \epsilon^2 E_3\end{bmatrix}=\begin{bmatrix} K_1^{(i+1)}-\bar{K}_1^{(i+1)} & \epsilon(K_2^{(i+1)}-\bar{K}_2^{(i+1)}) \\ \epsilon(K_2^{(i+1)}-\bar{K}_2^{(i+1)})^T & \epsilon(K_3^{(i+1)}-\bar{K}_3^{(i+1)})\end{bmatrix}. \tag{25}$$

Clearly, the $O(\epsilon^k)$ approximation of E will produce the $O(\epsilon^{k+1})$ approximation of the sought solution $K^{(i+1)}$, which is why we are interested in finding a convenient form for the error equation and an appropriate algorithm for its solution. It is shown in [12] and [13] that the error equation is given by

$$D_1^{(i)T}E_1+E_1D_1^{(i)}+D_3^{(i)T}E_2^T+E_2D_3^{(i)}=0 \tag{26a}$$

$$E_2D_4^{(i)}+\epsilon D_1^{(i)T}E_2+D_1^{(i)T}\bar{K}_2^{(i+1)}+D_3^{(i)T}E_3=0 \tag{26b}$$

$$E_3D_4^{(i)}+D_4^{(i)T}E_3+D_2^{(i)T}K_2^{(i+1)}+K_2^{(i+1)T}D_2^{(i)}+\epsilon(D_2^{(i)T}E_2+E_2^TD_2^{(i)})=0 \quad (26c)$$

and that the following algorithm

$$D_0^{(i)T}E_1^{(j+1)}+E_1^{(j+1)}D_0^{(i)}=D_0^{(i)T}(K_2^{(i+1)}+\epsilon E_2^{(j)})D_4^{(i)-1}D_3^{(i)}+D_3^{(i)T}D_4^{(i)-T}(K_2^{(i+1)}+\epsilon E_2^{(j)})^TD_0^{(i)} \quad (27a)$$

$$D_4^{(i)T}E_3^{(j+1)}+E_3^{(j+1)}D_4^{(i)}=-D_2^{(i)T}(K_2^{(i+1)}+\epsilon E_2^{(j)})-(K_2^{(i+1)}+\epsilon E_2^{(j)})^TD_2^{(i)} \quad (27b)$$

$$E_2^{(j+1)}=-D_3^{(i)T}E_3^{(j+1)}+E_1^{(j+1)}D_2^{(i)}+D_1^{(i)T}(K_2^{(i+1)}+\epsilon E_2^{(j)})D_4^{(i)-1} \qquad j=1,2,3,\cdots \quad (27c)$$

with initial conditions chosen as $E_1^{(0)}=0$, $E_2^{(0)}=0$, and $E_3^{(0)}=0$ converges to the required solution E with the rate of convergence of $O(\epsilon)$, that is,

$$\|E-E^{(j)}\|=O(\epsilon^j), \qquad j=1,2,3,\cdots. \quad (28)$$

That implies

$$\|K^{(i+1)}-(K^{(i+1)}+\epsilon E^{(j)})\|=O(\epsilon^j), \qquad j=1,2,3,\cdots. \quad (29)$$

Note that the complete slow-fast decomposition is achieved, that is, the solution of the Lyapunov equation (21) of order n_1+n_2 is obtained in terms of two low-order Lyapunov equations, the slow one (27a) of order n_1, and the fast one (27b) of order n_2.

Equation (20) is not a standard Lyapunov equation of singularly perturbed systems due to the fact that the initial conditions satisfy (17). In the following, we will apply the methodology of [11]-[13] to (20) subject to (17), and derive the recursive algorithm for its solution in terms of reduced order problems.

Setting $\epsilon=0$ in (20) will produce, after some algebra, the zero-order approximation of (20) as

$$L_1^{(i+1)}D_0^{(i)T}+D_0^{(i)}L_1^{(i+1)}+I=0 \quad (30a)$$

$$L_2^{(i+1)}=-L_1^{(i+1)}D_3^{(i)T}D_4^{(i)-T} \quad (30b)$$

$$L_3^{(i+1)}D_4^{(i)T}+D_4^{(i)}L_3^{(i+1)}+L_2^{(i+1)T}D_3^{(i)T}+D_3^{(i)}L_2^{(i+1)}=0. \quad (30c)$$

Even though the complete slow-fast decomposition is not achieved [contrary to (22)], these equations can be solved in terms of reduced order problems in a sequential manner, namely, first solve (30a), then (30b), and finally solve (30c).

Defining the error as

$$L^{(i+1)}-L^{(i+1)}=\epsilon M=\epsilon\begin{bmatrix}M_1 & M_2\\ M_2^T & M_3\end{bmatrix}=\begin{bmatrix}L_1^{(i+1)}-L_1^{(i+1)} & L_2^{(i+1)}-L_2^{(i+1)}\\ (L_2^{(i+1)}-L_2^{(i+1)})^T & L_3^{(i+1)}-L_3^{(i+1)}\end{bmatrix} \quad (31)$$

and subtracting (30) from (20), we get the error equation as

$$M_1D_1^{(i)T}+D_1^{(i)}M_1+D_2^{(i)}M_2^T+M_2D_2^{(i)T}=0 \quad (32a)$$

$$M_2D_4^{(i)T}+\epsilon D_1^{(i)}M_2+D_1^{(i)}L_2^{(i+1)}+D_2^{(i)}L_3^{(i+1)}+\epsilon D_2^{(i)}M_3+M_1D_3^{(i)T}=0 \quad (32b)$$

$$M_3D_4^{(i)T}+D_4^{(i)}M_3+D_3^{(i)}M_2+M_2^TD_3^{(i)T}+I=0. \quad (32c)$$

Note that (32b) is a weakly linear Lyapunov equation. At this point, we will ignore that fact and solve it with respect to M_2 as follows:

$$M_2=-[D_1^{(i)}(L_2^{(i+1)}+\epsilon M_2)+D_2^{(i)}(L_3^{(i+1)}+\epsilon M_3)+M_1D_3^{(i)T}]D_4^{(i)-T}. \quad (33)$$

Using (33) in (32a) yields

$$M_1D_0^{(i)T}+D_0^{(i)}M_1-D_2^{(i)}D_4^{(i)-1}H_2-H_2D_4^{(i)-T}D_2^{(i)T}=0 \quad (34)$$

where

$$H_2=D_1^{(i)}(L_2^{(i+1)}+\epsilon M_2)+D_2^{(i)}(L_3^{(i+1)}+\epsilon M_3)=D_1^{(i)}L_2^{(i+1)}+D_2^{(i)}L_3^{(i+1)}. \quad (35)$$

Thus, the weakly coupled and hierarchical structure of (32) can be exploited by proposing the following recursive scheme, which leads to the two low-order completely decoupled Lyapunov equations

$$M_1^{(j+1)}D_0^{(i)T}+D_0^{(i)}M_1^{(j+1)}-D_2^{(i)}D_4^{(i)-1}H_2^{(j)T}-H_2^{(j)}D_4^{(i)-T}D_2^{(i)T}=0 \quad (36a)$$

$$M_2^{(j+1)}=-[H^{(j)}+M_1^{(j+1)}D_3^{(i)T}]D_4^{(i)-T} \quad (36b)$$

$$M_3^{(j+1)}D_4^{(i)T}+D_4^{(i)}M_3^{(j+1)}+D_3^{(i)}M_2^{(j+1)}+M_2^{(j+1)T}D_3^{(i)T}+I=0 \quad (36c)$$

where

$$H^{(j)}=D_1^{(i)}(L_2^{(j+1)}+\epsilon M_2^{(j)})+D_2^{(i)}(L_3^{(j+1)}+\epsilon M_3^{(j)}) \quad (37)$$

with $j=0,1,2,3,\cdots$, with initial conditions chosen as $M_1^{(0)}=0$, $M_2^{(0)}=0$, and $M_3^{(0)}=0$.

The following theorem summarizes the features of the proposed scheme [17].

Theorem: The algorithm (36) converges, for sufficiently small values of ϵ, to the exact solution of the error terms, and thus to the solution $L^{(i+1)}$, with the rate of convergence of $O(\epsilon)$, that is,

$$\|M_k-M_k^{(j)}\|=O(\epsilon^j), \qquad k=1,2,3.$$

IV. Numerical Example

In order to demonstrate the efficiency of the proposed algorithm and the failure of the $O(\epsilon)$ theory, we have run a fifth-order real world example, and industrially important reactor [16]. Matrices A, B, C, Q, and R are given in [16]. The eigenvalues of the matrix A are -2.8, -7.7, -74, -82, -129. Thus, we have two slow and three fast variables. The small parameter ϵ is chosen as $\epsilon=0.1$, which is roughly the ratio of 7.7 and 74.

The theory of singularly perturbed optimal output feedback problems is derived so far for the $O(\epsilon)$ approximation. Using the $O(\epsilon)$ approximation of the equations comprising the solution of the optimal output feedback, namely, of (22) and (30), will fail to produce the desired approximation for this example. Even more, the algorithm does not converge to the near optimum solution for extremely small values of the parameter α such as 0.001. The cause of the trouble is the inversion of the quantity CLC^T. Its determinant for the optimal value of L is very small, that is, 0.9736×10^{-4}, and thus, this problem is very sensitive to $O(\epsilon)$ perturbations, which can be seen from Table I.

The results from Table I strongly support the necessity for the existence of the recursive schemes which can produce any desired accuracy, that is, the development of the $O(\epsilon^k)$ theory.

In Table II, we have presented results for the criterion $J_{opt}^{(i)}$ and the gain error for the global algorithm [5], and the corresponding quantities for the proposed reduced order recursive algorithm. The initial value for the gain $F^{(0)}$ is obtained from [4]. It can be seen that the initial guess is quite good, but the global algorithm converges very slowly to the optimal solution. As far as the criterion is concerned, it takes 28 iterations to achieve an accuracy of up to five decimal digits where $J_{opt}=0.28573$. On the other hand, the trajectories of the approximate system after 30 iterations are still far apart from the optimal trajectories since the approximate gain is only $O(10^{-2})$ close to the optimal one. Thus, this algorithm demands a lot of iterations in order to achieve high accuracy. This fact justifies even more the necessity for the existence of algorithms which will reduce computational requirements. In the proposed algorithm, only low-order Lyapunov equations are involved in algebraic computations. Even more, at the very beginning, they can be solved with reduced accuracy ($j=1$ or 2), and once we approach the optimum, the accuracy can be increased to the desired one. The third column of Table II is obtained for $j=2$ for $i\le16$, and $j=6$ for $i>16$. The second and fifth columns of Table II are obtained for $j=6$ for all i's. The parameter α is chosen as $\alpha=0.5$ since the global algorithm does not converge for $\alpha\ge0.6$.

TABLE I

α	$\det(C\mathcal{L}^{(1)}C^T)$ $j = 6$	$\det(C\mathcal{L}^{(1)}C^T)$ $j = 1$	$\det(C\underline{L}^{(0)}C^T)$	$\det(C\underline{L}^{(1)}C^T)$
0.5	0.86846×10^{-4}	0.14432×10^{-3}	0.38943×10^{-6}	0.24392×10^{-10}
0.1	0.12749×10^{-3}	"	"	0.57491×10^{-9}
0.01	0.14244×10^{-3}	"	"	0.31742×10^{-7}
0.001	0.14413×10^{-3}	"	"	0.24904×10^{-6}

TABLE II

$\epsilon=0.1$ $\alpha=0.5$ / i	$J^{(i)}_{opt}$	$J^{(i)}_{app}$ $j=6$	$J^{(i)}_{app}$ $j=2, i\le 16$ $j=6, i>16$	$\Vert F^{(i)}_{opt} - F_{opt}\Vert_\infty$	$\Vert F^{(i)}_{app} - F_{opt}\Vert_\infty$ $j=6$
1	0.30487	0.30488	0.30427	2.1520	2.1480
2	0.28733	0.28738	0.28879	0.1635	0.1684
4	0.28615	0.28619	0.28745	0.1296	0.1328
6	0.28595	0.28599	0.28710	0.1093	0.1120
8	0.28588	0.28591	0.28691	0.0913	0.0936
10	0.28583	0.28586	0.28676	0.0764	0.0783
12	0.28580	0.28583	0.28664	0.0638	0.0654
14	0.28578	0.28580	0.28654	0.0533	0.0550
16	0.28577	0.28578	0.28646	0.0446	0.0456
18	0.28575	0.28577	0.28584	0.0373	0.0380
20	0.28575	0.28576	0.28581	0.0311	0.0317
22	0.28574	0.28576	0.28579	0.0260	0.0256
24	0.28574	0.28575	0.28577	0.0217	0.0219
26	0.28574	0.28575	0.28577	0.0181	0.0181
28	0.28573	0.28575	0.28576	0.0150	0.0149
30	0.28573	0.28575	0.28575	0.0125	0.0122

IV. Conclusion

The reduced order numerical technique is obtained for the solution of the nonlinear algebraic equations of the output feedback control problem of singularly perturbed systems. It brings a considerable reduction in the size of required computations and makes the given problem numerically well defined. In addition, having obtained the reduced order problems, it can be easier to find a good initial guess $F^{(0)}$ and to handle the problem of nonuniqueness of the solution of (6)–(8)—they represent the necessary conditions only. A similar type of numerical technique can be developed for the other class of small parameter systems—the weakly coupled systems. Study in that direction is underway.

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On Optimal Control Law Implementations for Exponential Performance Index

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Abstract—**An alternate implementation is established, under certain conditions, for a previously obtained solution to the linear-exponential-Gaussian stochastic optimal control problem. This other implementation allows the current control to be specified without using future values of the measurement process parameters, which are often unavailable in practice.**

I. Introduction

Bensoussan and van Schuppen [1] have established a finite-dimensional solution for a type of linear-exponential-Gaussian optimal control problem, variants of which had been treated earlier by Jacobson [2], Speyer *et al.* [3], and Whittle [4]. But this solution, unlike the familiar one for the linear-quadratic-Gaussian case, depends on the use of future values of the measurement process parameters to specify the current control, which limits its value for applications. It is shown here that this dependence can be eliminated, at least within certain limits, by using a construction due to Speyer [5].

II. A "Linear-Exponential-Gaussian" Problem

The stochastic optimal control problem in question is over the time interval $T = [0, t_1]$, with the linear dynamics

$$dx = (Fx + Bu)dt + Gdw; \qquad x(0) \sim \text{normal } (\mu_0, P_0),$$

linear measurement process

$$dy = Hxdt + R^{1/2}db; \qquad y(0) = 0,$$

and exponential cost functional (to be minimized by the choice of control law generating u from past y values)

$$J = E\left\{\mu \exp\left(\frac{\mu}{2}\left[x^*(t_1)Mx(t_1) + \int_0^{t_1} (x^*Qx + u^*Nu)\, dt\right]\right)\right\}$$

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