# A New Filtering Method for Linear Singularly Perturbed Systems

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*Abstract*—In this paper we present a new method which allows complete decomposition of the optimal global Kalman filter for linear singularly perturbed systems into pure-slow and pure-fast local optimal filters both driven by the system measurements. The method is based on the exact decomposition of the global singularly perturbed algebraic Riccati equation into pure-slow and pure-fast local algebraic Riccati equations. An F-8 aircraft example demonstrates the proposed method.

## I. Filtering for Singularly Perturbed Linear Systems

Filtering problem of linear singularly perturbed continuous-time systems has been well documented in the control theory literature [1]–[7]. In [1]–[3] the suboptimal slow and fast Kalman filters were constructed producing an $O(\epsilon)$ accuracy for the estimates of the state trajectories, where a small positive singular perturbation parameter $\epsilon$ represents the separation between slow and fast phenomena. In [4]–[7] both the slow and fast (local) Kalman filters were obtained with an arbitrary order of accuracy, that is $O(\epsilon^k)$, where $k$ stands for either the number of terms of the Taylor series [4] or the number of the fixed-point iterations [5] used to calculate coefficients of the corresponding filters. It is important to point out that the local slow and fast filters in [4]–[5] are driven by the innovation process so that the additional communication channels are required to form the innovation process. In the newly proposed scheme, these filters will be driven by the system measurements only. In addition, the optimal filter gains will be completely determined in terms of the exact pure-slow and exact pure-fast reduced-order algebraic Riccati equations.

Consider the linear continuous-time invariant singularly perturbed system

$$\dot{x}_1 = A_1 x_1 + A_2 x_2 + G_1 w_1$$
$$\epsilon \dot{x}_2 = A_3 x_1 + A_4 x_2 + G_2 w_1 \tag{1}$$

with the corresponding measurements

$$y = C_1 x_1 + C_2 x_2 + w_2 \tag{2}$$

where $x_1 \in R^{n_1}$ and $x_2 \in R^{n_2}$ are state vectors, $w_1 \in R^r$ and $w_2 \in R^1$ are zero-mean stationary, white Gaussian noise stochastic processes with intensities $W_1 > 0$ and $W_2 > 0$, respectively, and $y \in R^1$ are the system measurements. In the following $A_i$, $G_j$, $C_j$, $i = 1, 2, 3, 4$, $j = 1, 2$, are constant matrices. We assume that the system under consideration has the standard singularly perturbed form, [10], that is, the following assumption is satisfied.

*Assumption 1:* The fast subsystem matrix $A_4$ is nonsingular.

The optimal Kalman filter, corresponding to (1)–(2), driven by the innovation process is given by

$$\dot{\hat{x}}_1 = A_1 \hat{x}_1 + A_2 \hat{x}_2 + K_1 v$$
$$\epsilon \dot{\hat{x}}_2 = A_3 \hat{x}_1 + A_4 \hat{x}_2 + K_2 v$$
$$v = y - C_1 \hat{x}_1 - C_2 \hat{x}_2 \tag{3}$$

where the optimal filter gains $K_1$ and $K_2$ are obtained from [4]

$$K_1 = (P_1 C_1^T + P_2 C_2^T) W_2^{-1},$$
$$K_2 = (\epsilon P_2^T C_1^T + P_3 C_2^T) W_2^{-1} \tag{4}$$

with matrices $P_1$, $P_2$, and $P_3$ representing the positive semidefinite stabilizing solution of the filter algebraic Riccati equation

$$AP + PA^T - PSP + GW_1 G^T = 0 \tag{5}$$

where

$$A = \begin{bmatrix} A_1 & A_2 \\ \frac{A_3}{\epsilon} & \frac{A_4}{\epsilon} \end{bmatrix}, \quad G = \begin{bmatrix} G_1 \\ \frac{G_2}{\epsilon} \end{bmatrix},$$

$$S = C^T W_2^{-1} C, \quad P = \begin{bmatrix} P_1 & P_2 \\ P_2^T & \frac{1}{\epsilon} P_3 \end{bmatrix}. \tag{6}$$

For the decomposition and approximation of the singularly perturbed Kalman filter (3) the Chang transformation [9] has been used in [4]–[5]

$$\begin{bmatrix} \hat{\eta}_1 \\ \hat{\eta}_2 \end{bmatrix} = \begin{bmatrix} I - \epsilon HL & -\epsilon H \\ L & I \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} \tag{7}$$

where $L$ and $H$ satisfy algebraic equations

$$A_4 L - A_3 - \epsilon L(A_1 - A_2 L) = 0$$

$$-H A_4 + A_2 - \epsilon HLA_2 + \epsilon(A_1 - A_2 L)H = 0. \tag{8}$$

The Chang transformation applied to (3) produces

$$\dot{\hat{\eta}}_1 = (A_1 - A_2 L)\hat{\eta}_1 + (K_1 - HK_2 - \epsilon HLK_1)v$$

$$\epsilon \dot{\hat{\eta}}_2 = (A_4 + \epsilon LA_2)\hat{\eta}_2 + (K_2 + \epsilon LK_1)v. \tag{9}$$

In the new coordinates the innovation process is given by

$$v = y - (C_1 - C_2 L)\hat{\eta}_1 - [C_2 + \epsilon(C_1 - C_2 L)H]\hat{\eta}_2. \tag{10}$$

In [4]–[5], the approximate reduced-order filters of (9)–(10) were defined as well.

Equation (8) is solvable and produces the unique solutions under Assumption 1. Equation (5) produces the unique stabilizing solutions under the following assumption.

*Assumption 2:* The subsystem matrices in the algebraic Riccati equation (5)–(6) satisfy the standard stabilizability-detectability conditions, [4]–[7].

## II. A New Method for Filter Decomposition

In the decomposition procedure from the previous section, the slow and fast filters (9) require some additional communication channels necessary to form the innovation process (10); see Fig. 1. Here, we propose a new decomposition scheme such that the slow and fast filters are completely decoupled and both of them are driven by the system measurements. The new method is based on the pure-slow pure-fast decomposition technique for solving the regulator algebraic Riccati equation of singularly perturbed systems [8]. We give an
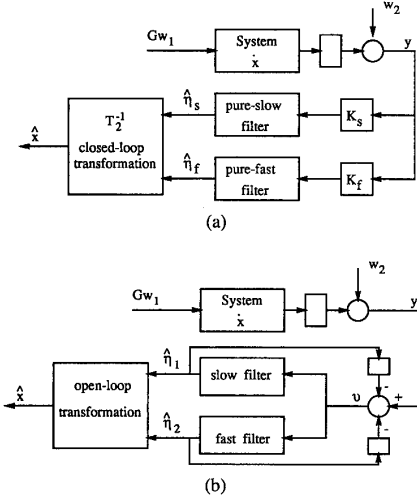
Fig. 1. (a) New filtering method; (b) old filtering method.

additional interpretation of the results of [8] which will be used in this paper.

Consider the linear-quadratic optimal control problem of (1), that is

$$\dot{x}_1 = A_1 x_1 + A_2 x_2 + B_1 u$$

$$\epsilon \dot{x}_2 = A_3 x_1 + A_4 x_2 + B_2 u$$

$$J = \int_0^\infty \left[ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T Q \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + u^T R u \right] dt, \quad Q \geq 0, \quad R > 0 \quad (11)$$

where the control vector, $u \in R^m$, has to be chosen such that the performance criterion, $J$, is minimized. The very well-known solution to this problem is given by

$$u = -R^{-1} B^T P_r x = -F_1 x_1 - F_2 x_2 \quad (12)$$

where $P_r$ is the positive semidefinite solution of the regulator algebraic Riccati equation

$$A^T P_r + P_r A + Q - P_r Z P_r = 0 \quad (13)$$

with

$$Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix}, \quad Z = BR^{-1}B^T,$$

$$B = \begin{bmatrix} B_1 \\ \frac{B_2}{\epsilon} \end{bmatrix}, \quad P_r = \begin{bmatrix} P_{1r} & \epsilon P_{2r} \\ \epsilon P_{2r}^T & \epsilon P_{3r} \end{bmatrix}. \quad (14)$$

The optimal regulator gains $F_1$ and $F_2$ are given by

$$F_1 = R^{-1}(B_1^T P_{1r} + B_2^T P_{2r}^T),$$

$$F_2 = R^{-1}(\epsilon B_1^T P_{2r} + B_2^T P_{3r}). \quad (15)$$

The results of interest that we need, which can be deduced from [8] are given in the form of the following lemma.

*Lemma 1:* Consider the optimal closed-loop linear system

$$\dot{x}_1 = (A_1 - B_1 F_1)x_1 + (A_2 - B_1 F_2)x_2$$

$$\epsilon \dot{x}_2 = (A_3 - B_2 F_1)x_1 + (A_4 - B_2 F_2)x_2 \quad (16)$$

then there exists a nonsingular transformation **T**

$$\begin{bmatrix} \xi_s \\ \xi_f \end{bmatrix} = \mathbf{T} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (17)$$

such that

$$\dot{\xi}_s = (a_1 + a_2 P_{rs})\xi_s$$

$$\epsilon \dot{\xi}_f = (b_1 + b_2 P_{rf})\xi_f \quad (18)$$

where $P_{rs}$ and $P_{rf}$ are the unique solutions of the exact pure-slow and pure-fast completely decoupled algebraic Riccati equations

$$0 = P_{rs}a_1 - a_4 P_{rs} - a_3 + P_{rs}a_2 P_{rs}$$

$$0 = P_{rf}b_1 - b_4 P_{rf} - b_3 + P_{rf}b_2 P_{rf}. \quad (19)$$

Matrices $a_i$, $b_i$, $i = 1, 2, 3, 4$, can be found in [8]. In this paper we will give their expressions for the corresponding filter pure-slow and pure-fast algebraic Riccati equations to be defined later. The nonsingular transformation **T** is given by

$$\mathbf{T} = (\Pi_1 + \Pi_2 P_r). \quad (20)$$

Even more, the global solution $P_r$ can be obtained from the reduced-order exact pure-slow and pure-fast algebraic Riccati equations, that is

$$P_r = \left( \Omega_3 + \Omega_4 \begin{bmatrix} P_{rs} & 0 \\ 0 & P_{rf} \end{bmatrix} \right) \left( \Omega_1 + \Omega_2 \begin{bmatrix} P_{rs} & 0 \\ 0 & P_{rf} \end{bmatrix} \right)^{-1}. \quad (21)$$

Known matrices $\Omega_i$, $i = 1, 2, 3, 4$, and $\Pi_1, \Pi_2$ are given in terms of the solutions of the Chang decoupling equations [8].

$$\triangle$$

The desired slow-fast decomposition of the Kalman filter (3) will be obtained by producing a dual lemma to Lemma 1. Consider the optimal closed-loop Kalman filter (3) driven by the system measurements, that is

$$\dot{\hat{x}}_1 = (A_1 - K_1 C_1)\hat{x}_1 + (A_2 - K_1 C_2)\hat{x}_2 + K_1 y$$

$$\epsilon \dot{\hat{x}}_2 = (A_3 - K_2 C_1)\hat{x}_1 + (A_4 - K_2 C_2)\hat{x}_2 + K_2 y \quad (22)$$

with the optimal filter gains $K_1$ and $K_2$ calculated from (4)–(6). By duality between the optimal filter and regulator, the filter Riccati equation (5) can be solved by using the same decomposition method for solving (13) with

$$A \rightarrow A^T, \quad Q \rightarrow GW_1 G^T, \quad F^T = K$$

$$Z = BR^{-1}B^T \rightarrow S = C^T W_2^{-1} C. \quad (23)$$

By invoking results from [8], and using duality, the following matrices have to be formed (see also [5])

$$T_1 = \begin{bmatrix} A_1^T & -C_1^T W_2^{-1} C_1 \\ -G_1 W_1 G_1^T & -A_1 \end{bmatrix},$$

$$T_2 = \begin{bmatrix} A_3^T & -C_1^T W_2^{-1} C_2 \\ -G_1 W_1 G_2^T & -A_2 \end{bmatrix},$$

$$T_3 = \begin{bmatrix} A_2^T & -C_2^T W_2^{-1} C_1 \\ -G_2 W_1 G_1^T & -A_3 \end{bmatrix},$$

$$T_4 = \begin{bmatrix} A_4^T & -C_2^T W_2^{-1} C_2 \\ -G_2 W_1 G_2^T & -A_4 \end{bmatrix}. \quad (24)$$

Note that on the contrary to the results from [8] where the state-costate variables have to be partitioned as $x = [x_1^T x_2^T]^T$ and $p = [p_1^T \epsilon p_2^T]^T$, in the case of the dual filter variables, we have to use the following partitions $x = [x_1^T \epsilon x_2^T]^T$ and $p = [p_1^T p_2^T]^T$. Since matrices $T_1$, $T_2$, $T_3$, $T_4$ correspond to the system matrices of a singularly perturbed linear system, the slow-fast decomposition is achieved by using the Chang decoupling equations

$$T_4 M - T_3 - \epsilon M(T_1 - T_2 M) = 0$$

$$-N(T_4 + \epsilon M T_2) + T_2 + \epsilon(T_1 - T_2 M)N = 0. \qquad (25)$$

By using the permutation matrices dual to those from [8], (note $E_1$ is different than the corresponding one from [8])

$$E_1 = \begin{bmatrix} I_{n1} & 0 & 0 & 0 \\ 0 & 0 & I_{n1} & 0 \\ 0 & \frac{I_{n2}}{\epsilon} & 0 & 0 \\ 0 & 0 & 0 & I_{n2} \end{bmatrix},$$

$$E_2 = \begin{bmatrix} I_{n1} & 0 & 0 & 0 \\ 0 & 0 & I_{n1} & 0 \\ 0 & I_{n2} & 0 & 0 \\ 0 & 0 & 0 & I_{n2} \end{bmatrix} \qquad (26)$$

we can define

$$\Pi = \begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_3 & \Pi_4 \end{bmatrix} = E_2^T \begin{bmatrix} I - \epsilon NM & -\epsilon N \\ M & I \end{bmatrix} E_1. \qquad (27)$$

Then, the desired transformation is given by

$$\mathbf{T}_2 = (\Pi_1 + \Pi_2 P). \qquad (28)$$

The transformation $\mathbf{T}_2$ applied to the filter variables as

$$\begin{bmatrix} \hat{\eta}_s \\ \hat{\eta}_f \end{bmatrix} = \mathbf{T}_2^{-T} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} \qquad (29)$$

produces

$$\begin{bmatrix} \dot{\hat{\eta}}_s \\ \dot{\hat{\eta}}_f \end{bmatrix} = \mathbf{T}_2^{-T} \begin{bmatrix} A_1 - K_1 C_1 & A_2 - K_1 C_2 \\ \frac{A_3 - K_2 C_1}{\epsilon} & \frac{A_4 - K_2 C_2}{\epsilon} \end{bmatrix}$$
$$\cdot \mathbf{T}_2^T \begin{bmatrix} \hat{\eta}_s \\ \hat{\eta}_f \end{bmatrix} + \mathbf{T}_2^{-T} \begin{bmatrix} K_1 \\ \frac{K_2}{\epsilon} \end{bmatrix} y \qquad (30)$$

such that the complete closed-loop decomposition is achieved, that is

$$\dot{\hat{\eta}}_s = (a_1 + a_2 P_s)^T \hat{\eta}_s + K_s y$$

$$\epsilon \dot{\hat{\eta}}_f = (b_1 + b_2 P_f)^T \hat{\eta}_f + K_f y. \qquad (31)$$

The matrices in (31) are given by

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = (T_1 - T_2 M), \quad \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = (T_4 + \epsilon M T_2) \qquad (32)$$

$$\begin{bmatrix} K_s \\ \frac{K_f}{\epsilon} \end{bmatrix} = \mathbf{T}_2^{-T} \begin{bmatrix} K_1 \\ \frac{K_2}{\epsilon} \end{bmatrix}$$

$$0 = P_s a_1 - a_4 P_s - a_3 + P_s a_2 P_s$$

$$0 = P_f b_1 - b_4 P_f - b_3 + P_f b_2 P_f. \qquad (33)$$

A method for solving nonsymmetric Riccati equations (33) can be found in [8]. It is important to point out that the matrix $P$ in (28) can be obtained in terms of $P_s$ and $P_f$ by using formula (21) with

$$P_{rs} = P_s, \quad P_{rf} = P_f \qquad (34)$$

and $\Omega_1$, $\Omega_2$, $\Omega_3$, $\Omega_4$ obtained from

$$\Omega = \begin{bmatrix} \Omega_1 & \Omega_2 \\ \Omega_3 & \Omega_4 \end{bmatrix} = E_1^{-1} \begin{bmatrix} I & \epsilon N \\ -M & I - \epsilon MN \end{bmatrix} E_2^{-T}. \qquad (35)$$

A lemma dual to Lemma 1 can be now formulated.

*Lemma 2:* Given the closed-loop optimal Kalman filter (22) of a linear singularly perturbed system. Then there exists a nonsingular transformation matrix (28), which completely decouples (22) into pure-slow and pure-fast local filters (31) both driven by the system measurements. Even more, the decoupling transformation (28) and the filter coefficients given in (32) can be obtained in terms of the exact pure-slow and pure-fast reduced-order completely decoupled Riccati equations (33). △

We can now define the corresponding approximations (in the spirit of the theory of singular perturbations, [4]–[6]) of the pure-slow and pure-fast filters as

$$\dot{\hat{\eta}}_s^{(k)} = (a_1^{(k)} + a_2^{(k)} P_s^{(k)})^T \hat{\eta}_s^{(k)} + K_s^{(k)} y$$
$$\epsilon \dot{\hat{\eta}}_f^{(k)} = (b_1^{(k)} + b_2^{(k)} P_f^{(k)})^T \hat{\eta}_f^{(k)} + K_f^{(k)} y \qquad (36)$$

where

$$\begin{bmatrix} a_1^{(k)} & a_2^{(k)} \\ a_3^{(k)} & a_4^{(k)} \end{bmatrix} = (T_1^{(k)} - T_2^{(k)} M^{(k)})$$

$$\begin{bmatrix} b_1^{(k)} & b_2^{(k)} \\ b_3^{(k)} & b_4^{(k)} \end{bmatrix} = (T_4^{(k)} + \epsilon M^{(k-1)} T_2^{(k-1)})$$

$$P_s^{(k)} = P_s + O(\epsilon^k), \quad P_f^{(k)} = P_f + O(\epsilon^k),$$

$$M^{(k)} = M + O(\epsilon^k)$$

$$\begin{bmatrix} K_s^{(k)} \\ \frac{K_f^{(k)}}{\epsilon} \end{bmatrix} = \mathbf{T}_2^{(k)-T} \begin{bmatrix} K_1^{(k)} \\ \frac{K_2^{(k)}}{\epsilon} \end{bmatrix}. \qquad (37)$$

Note that in the expression for $b_i^{(k)}$ we can use $M^{(k-1)}$ and $T_2^{(k-1)}$ since these matrices are multiplied by $\epsilon$ so that we get $b_i = b_i^{(k)} + O(\epsilon^k)$.

A comparison between the new filtering method and the one already known for linear singularly perturbed systems is given in Fig. 1. It can be seen that the new filtering method allows complete decomposition and parallelism between pure-slow and pure-fast filters.

## III. AN F-8 AIRCRAFT EXAMPLE

To demonstrate the proposed method we solve the same aircraft example as the one done in [4]–[5]. The problem matrices are given by [5]

$$A_1 = \begin{bmatrix} 0.278386 & -0.965256 \\ 0.089833 & -0.290700 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -0.074210 & 0.016017 \\ 0.012815 & -0.001398 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -0.001815 & 0.005873 \\ 0.002850 & -0.009223 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} -0.030344 & 0.075024 \\ -0.075092 & -0.016777 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0 & 0 \\ 1 & -3.236 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 0.00500 \\ -0.003152 & 0.01302 \end{bmatrix}$$

$$G_1 = \begin{bmatrix} -46.62696 \\ 7.858776 \end{bmatrix}, \quad G_2 = \begin{bmatrix} -18.210002 \\ -45.049998 \end{bmatrix}$$

$$W_1 = 0.000315, \quad W_2 = \text{diag}[0.000686 \quad 40], \quad \epsilon = 0.025.$$

We have obtained completely decoupled filters driven by the measurements $y$ as

$$\dot{\hat{\eta}}_s = \begin{bmatrix} 0.2755 & -0.9558 \\ 0.0903 & -0.2923 \end{bmatrix} \hat{\eta}_s + \begin{bmatrix} -0.2561 & 0.0018 \\ -0.0958 & 0.0000 \end{bmatrix} y$$

$$\epsilon \dot{\hat{\eta}}_f = \begin{bmatrix} -1.2151 & 1.1831 \\ -2.9733 & -5.1789 \end{bmatrix} \hat{\eta}_f + \begin{bmatrix} 9.1085 & 0.0028 \\ 22.5077 & 0.0039 \end{bmatrix} y.$$

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# Strong Stabilizability of Systems with Multiaffine Uncertainties and Numerator Denominator Coupling

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*Abstract*—This paper considers a set of proper transfer functions whose numerator and denominator polynomial coefficients display dependent multiaffine parametric uncertainties. It is shown that provided no member transfer function has positive real pole-zero cancellations, all members satisfy the parity interlacing property iff all corner members do the same. Notice that while this implies that each member is strongly stabilizable, it does not imply the existence of a single stable controller that stabilizes the whole set. The paper also shows that whenever the numerator and denominator polynomials lie in dependent polytopes, the task of verifying the absence of positive real pole-zero cancellations can be accomplished by checking the edges.

## I. INTRODUCTION

An important question in Linear Systems Theory concerns the conditions under which a plant is strongly stabilizable, i.e., can be stabilized through a stable proper compensator [1], [2]. It is known [1], [2], that proper plants are strongly stabilizable iff: a) they have no unstable pole-zero cancellations and b) they obey the so-called parity interlacing property (pip) defined below.

*Definition 1.1:* A rational transfer function obeys pip if between every two positive real zeros (including those at infinity) of this transfer function, there are an even number of positive real poles (multiplicity included).

This paper concerns the verification of pip for a family of proper transfer functions. The significance of this problem to adaptive and robust control has been discussed in [3]. One of the questions posed and answered in [3] concerns a family of strictly proper transfer functions whose numerator and denominator lie in independent polytopes. Under the assumption that each member of the family is free from unstable pole-zero cancellations, [3] shows that the family is pip invariant, i.e., all its members are pip iff the ratio of each numerator corner with each denominator corner is pip. At the same time, [3] also demonstrates through a counterexample that this corner result fails to extend to the biproper case.

The present paper considers the same question as that addressed in [3], but with respect to a parameterization that enjoys a much wider range of applicability than what is considered in [3]. Specifically, we consider here the family of transfer functions defined in (1.1)–(1.4)

$$T(s, K) = \left\{ t(s, k) = \frac{p(s, k)}{q(s, k)} | \forall k \in K, \ p(s, k) \in P(s, k), \right.$$

$$\left. q(s, k) \in Q(s, k) \right\}, \quad (1.1)$$

$$P(s, K) = \left\{ p(s, k) = s^n + \sum_{i=1}^{n} p_i(k) s^{n-i} | \forall k \in K \right\}, \quad (1.2)$$

$$Q(s, K) = \left\{ q(s, k) = s^m + \sum_{i=1}^{m} q_i(k) s^{m-i} | \forall k \in K \right\} \quad (1.3)$$