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Lyapunov Iterations for Optimal Control of Jump Linear Systems at Steady State

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***Abstract*—In this paper we construct a sequence of Lyapunov algebraic equations whose solutions converge to the solutions of the coupled algebraic Riccati equations of the optimal control problem for jump linear systems. The obtained solutions are positive semidefinite, stabilizing, and unique. The proposed algorithm is extremely efficient from the numerical point of view since it operates only on the reduced-order decoupled Lyapunov equations. Several examples are included to demonstrate the procedure.**

I. Introduction

Systems of coupled Riccati equations occur in several classes of optimal control problems such as jump linear control systems [1]. In this class of control problems, a set of strongly coupled Riccati equations is to be solved to determine the optimal feedback gains. A homotopy algorithm for solving systems of coupled differential Riccati equations of jump linear systems is presented in [2]. This method is computationally expensive and, hence, undesirable for solving algebraic equations of the corresponding steady-state problem. In [10], a method based on successive approximations led to the problem of solving a set of coupled differential Lyapunov equations. It can be easily observed that at steady state the method of [10] is in fact the Newton method for solving the corresponding algebraic equations. The Newton algorithm given in terms of the coupled algebraic Lyapunov equations is also presented in [3]; however, while it is known for fast convergence, it suffers from its strong dependence on the proximity of the initial guess to the actual solution.

In this paper, we introduce a new algorithm for solving coupled algebraic Riccati equations of jump linear systems that converges to the optimal solution regardless of the proximity of the initial guess to the actual solution in a relatively small number of iterations. In addition, the algorithm is extremely efficient from the computational point of view since it operates only on the reduced-order decoupled algebraic Lyapunov equations.

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II. Problem Formulation

Consider a linear dynamic system described by

$$\dot{x}(t) = A(r)x(t) + B(r)u(t), \qquad x(t_0) = x_0 \tag{1}$$

where $x(t)$ is n-dimensional vector of the system states, $u(t)$ is a control input of dimension m, A and B are mode-dependent matrices of appropriate dimensions, and r is a Markovian random process that represents the mode of the system and takes on values in a discrete set $\Psi = \{1, 2, \cdots, N\}$. The stationary transition probabilities of the modes of the system are determined by the transition rate matrix given by

$$\Pi = \begin{bmatrix} \pi_{11} & \pi_{12} & \dots & \pi_{1N} \\ \pi_{21} & \pi_{22} & \dots & \pi_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ \pi_{N1} & \pi_{N2} & \dots & \pi_{NN} \end{bmatrix}. \tag{2}$$

The matrix Π has the property that $\pi_{ij} \geq 0, i \neq j$ and $\pi_{ii} = -\sum_{j\neq i} \pi_{ij}$, [4]. The performance of system (1) is evaluated by the criterion

$$J = E\left\{ \int_0^\infty \left[x^T(t)Q(r)x(t) + u^T(t)R(r)u(t)\right]dt \middle| t_0, x(t_0), r(t_0) \right\} \tag{3}$$

where $Q(r) \geq 0$ and $R(r) > 0$ for every r. The optimal feedback controls of (1)–(3) are given by [1]

$$u_{opt}(t) = -R_k^{-1}B_k^T P_k x(t), \quad k = 1, 2, \cdots, N \tag{4}$$

where the subscript k indicates that the system is in mode $r = k$, that is

$$\begin{aligned} A(r = k) &= A_k, \qquad B(r = k) = B_k \\ Q(r = k) &= Q_k, \qquad R(r = k) = R_k \end{aligned} \tag{5}$$

and $P_k^{'}s$, $k = 1, 2, \cdots, N$, are the positive semidefinite stabilizing solutions of a set of the coupled algebraic Riccati equations

$$\mathbf{A_k^T} P_k + P_k \mathbf{A_k} - P_k S_k P_k + Q_k + \sum_{j=1, j\neq k}^{N} \pi_{kj} P_j = 0, k = 1, 2, \cdots, N \tag{6}$$

where

$$\mathbf{A_k} = A_k + \frac{1}{2}\pi_{kk} I, \qquad S_k = B_k R_k^{-1} B_k^T. \tag{7}$$

Equations (6)–(7) are nonlinear algebraic equations. The existence of positive semidefinite stabilizing solutions (stabilizable with respect to $\mathbf{A_k}$) of these equations is established in [10] (see also [1] and [8]) under the following assumption.

Assumption 1: The triples $\left(A_i, B_i, \sqrt{Q_i}\right)$, $i = 1, 2, , \cdots, N$ are stabilizable-detectable and

$$\max_{i=1,\cdots,N} \left\{ \inf_{\Gamma_i} \left| \lambda_{max} \left[\int_0^\infty e^{\left(A_i+B_i\Gamma_i+\frac{1}{2}\pi_{ii}I\right)^T t} \times e^{\left(A_i+B_i\Gamma_i+\frac{1}{2}\pi_{ii}I\right)t} dt \right] \right| \right\} < 1 \tag{8}$$

where Γ_i are arbitrary real matrices.

Condition (8) is a consequence of the fixed point iterations [8], [10] used to establish the existence and uniqueness of the solutions of (6)–(7). It is "crucial...and its conservativeness is difficult to evaluate" [1]. Note that in [11] the notion of stochastic stabilizability is introduced to replace both condition (8) and deterministic stabilizability of the pairs (A_i, B_i). It is important to notice that Assumption 1 guarantees the stabilizability of the matrices $\mathbf{A_k}$ by the optimal closed-loop feedback gains. It is very natural to assume that the optimal feedback controls stabilize the actual systems, [1], that is, to impose the additional assumption.

Assumption 2: The system matrices A_k, $k = 1, 2, \cdots, N$ are stabilizable by the optimal feedback controls (4).

III. Main Result

Assume that all conditions in Assumption 1 are satisfied, that is, assume that the unique stabilizing $P_k \geq 0$, $k = 1, 2, \cdots, N$ exist. We propose the following algorithm (in terms of decoupled algebraic Lyapunov equations) for solving the set of coupled algebraic Riccati equations (6)–(7).

Algorithm:

$$\begin{aligned} \left(\mathbf{A_k} - S_k P_k^{(i)}\right)^T P_k^{(i+1)} + P_k^{(i+1)} \left(\mathbf{A_k} - S_k P_k^{(i)}\right) \\ = -P_k^{(i)} S_k P_k^{(i)} - Q_k^{(i)} \\ \text{with stabilizing } P_k^{(0)} \geq 0, k = 1, 2, \cdots, N \end{aligned} \tag{9}$$

where

$$Q_k^{(i)} = Q_k + \sum_{j=1, j\neq k}^{N} \pi_{kj} P_j^{(i)} \geq 0. \tag{10}$$

Thus, the solution of the $n \times N$th order nonlinear coupled algebraic Riccati equations will be obtained by performing iterations on N decoupled linear algebraic Lyapunov equations each of order n.

Note that by (7) and Assumption 1, the triples $\left(\mathbf{A_k}, B_k, \sqrt{Q_k}\right)$ are stabilizable-detectable. In the following it will be shown that each sequence of solutions of (9)–(10) is nested between two sequences

$$K_k^{(i)} \leq P_k^{(i)} \leq \overline{P_k^{(i)}}, \qquad \forall i, \ \forall k \tag{11}$$

with $\left\{K_k^{(i)}\right\}$ monotonically converging to P_k from below, that is

$$K_k^{(0)} \leq K_k^{(1)} \leq \cdots . \leq P_k, \qquad k = 1, 2, \cdots, N \tag{12}$$

and the sequence $\left\{\overline{P_k^{(i)}}\right\}$ monotonically converging to P_k from above, that is

$$\overline{P_k^{(0)}} \geq \overline{P_k^{(1)}} \geq \cdots . \geq P_k. \tag{13}$$

Proof of Convergence

Lower Bounds: Consider the following sequences of the standard algebraic Riccati equations

$$\begin{aligned} &\mathbf{A_k^T} K_k^{(i+1)} + K_k^{(i+1)} \mathbf{A_k} - K_k^{(i+1)} S_k K_k^{(i+1)} + \mathbf{Q_k^{(i)}} = 0 \\ &\mathbf{Q_k^{(i)}} = Q_k + \sum_{j=1, j\neq k}^{N} \pi_{kj} K_j^{(i)}, \quad \text{with} \quad K_k^{(0)} = 0, \ \forall k. \end{aligned} \tag{14}$$

Note that by Assumption 1 the unique positive semidefinite stabilizing solutions of (14) exist for each iteration index i. For $i = 0$, we have

$$\mathbf{A_k^T} K_k^{(1)} + K_k^{(1)} \mathbf{A_k} - K_k^{(1)} S_k K_k^{(1)} + Q_k = 0, \quad k = 1, 2, \cdots, N. \tag{15}$$

By Assumption 1 the required positive semidefinite stabilizing solutions $P_k, k = 1, 2, \cdots, N$ of (6) exist. Using the known results on comparison of the solutions for the standard algebraic Riccati equations, (for example, [6]–[7]), it follows from (6) and (15) that

$$Q_k \leq Q_k + \sum_{j=1, j\neq k}^{N} \pi_{kj} P_j \Rightarrow P_k \geq K_k^{(1)}. \tag{16}$$

For $i = 1$, we have

$$\begin{aligned} \mathbf{A_k^T} K_k^{(2)} + K_k^{(2)} \mathbf{A_k} - K_k^{(2)} S_k K_k^{(2)} + Q_k \\ + \sum_{j=1, j\neq k}^{N} \pi_{kj} K_j^{(1)} = 0, \qquad k = 1, 2, \cdots, N \end{aligned} \tag{17}$$

and since by (16)

$$Q_k + \sum_{j=1, j\neq k}^{N} \pi_{kj} K_j^{(1)} \leq Q_k + \sum_{j=1, j\neq k}^{N} \pi_{kj} P_j \tag{18}$$

it follows that $P_k \geq K_k^{(2)}$. Also, due to the fact that

$$Q_k \leq Q_k + \sum_{j=1, j\neq k}^{N} \pi_{kj} K_j^{(1)} \tag{19}$$

we have $K_k^{(2)} \geq K_k^{(1)}$. Continuing the same procedure, we get from (14) monotonically nondecreasing sequences of positive semidefinite matrices bounded above by the solutions of (6), that is, by P_k

$$0 = K_k^{(0)} \leq K_k^{(1)} \leq K_k^{(2)} \leq \cdots \leq P_k, \quad k = 1, 2, \cdots, N. \tag{20}$$

These sequences are convergent and their limit points are $P_k, k = 1, 2, \cdots, N$ [8]–[9].

Remark: Note that the recursive scheme (14), given in terms of the standard Riccati equations, can be used also for numerical solution of (6)–(7).

Now we show that the sequences of positive semidefinite matrices generated by the proposed algorithm, $\left\{P_k^{(i)}\right\}$, are bounded from below by the sequences of positive semidefinite matrices generated by solving (14), that is, by $\left\{K_k^{(i)}\right\}$. From (9)–(10) and (14) we get

$$\begin{aligned} &\left(\mathbf{A_k} - S_k P_k^{(i)}\right)^T \left(P_k^{(i+1)} - K_k^{(i+1)}\right) \\ &+ \left(P_k^{(i+1)} - K_k^{(i+1)}\right)\left(\mathbf{A_k} - S_k P_k^{(i)}\right) = \\ &- \sum_{j=1, j\neq k}^{N} \pi_{kj} \left(P_j^{(i)} - K_j^{(i)}\right) \\ &- \left(P_k^{(i)} - K_k^{(i+1)}\right) S_k \left(P_k^{(i)} - K_k^{(i+1)}\right) \\ &k = 1, 2, \cdots, N; \ i = 1, 2, 3, \cdots. \end{aligned} \tag{21}$$

Since $K_k^{(0)} = 0$ and $P_k^{(0)} \geq 0$, $\forall k \Rightarrow P_k^{(0)} - K_k^{(0)} \geq 0$, it follows that the right-hand side of (21) is negative semidefinite for $i = 0$ so that we have $P_k^{(1)} - K_k^{(1)} \geq 0,\ \forall k$. To establish that the matrices $\mathbf{A_k} - S_k P_k^{(i)}$ are stable for every k and i, we apply the stability proof technique from [6, pp. 1326–1327] to our problem. Rewrite (21) in the form

$$\begin{aligned}&\left(\mathbf{A_k} - S_k P_k^{(i+1)}\right)^T \left(P_k^{(i+1)} - K_k^{(i+1)}\right)\\&+ \left(P_k^{(i+1)} - K_k^{(i+1)}\right)\left(\mathbf{A_k} - S_k P_k^{(i+1)}\right) =\\&- \sum_{j=1, j\neq k}^{N} \pi_{kj}\left(P_j^{(i)} - K_j^{(i)}\right)\\&- \left(P_k^{(i+1)} - K_k^{(i+1)}\right)S_k\left(P_k^{(i+1)} - K_k^{(i+1)}\right)\\&- \left(P_k^{(i+1)} - P_k^{(i)}\right)S_k\left(P_k^{(i+1)} - P_k^{(i)}\right),\\&k = 1, 2, \cdots, N;\ i = 1, 2, 3, \cdots. \qquad (22)\end{aligned}$$

The required stability proof technique from [6] is done by contradiction. Let us assume that the matrices $\mathbf{A_k} - S_k P_k^{(0)}$ are stable, but the matrix $\mathbf{A_k} - S_k P_k^{(1)}$ is unstable for some k. Then, there exists an eigenvalue λ such that

$$\exists k \text{ such that } \left(A_k - S_k P_k^{(1)}\right)x = \lambda x,\quad x \neq 0,\quad Re\{\lambda\} \geq 0. \qquad (23)$$

Using (23) in (22) we get

$$2x^T Re\{\lambda\}\left(P_k^{(1)} - K_k^{(1)}\right)x = -x^T M_k^{(1)} x,\qquad M_k^{(1)} \geq 0, \text{ for some } k. \qquad (24)$$

Since the left-hand side of (24) is positive semidefinite, the equality in (24) is valid only for $x^T M_k^{(1)} x = 0$. From (22) we have

$$\begin{aligned}x^T M_k^{(1)} x = x^T[&\left(P_k^{(1)} - P_k^{(0)}\right)S_k\left(P_k^{(1)} - P_k^{(0)}\right)\\&+ \left(P_k^{(1)} - K_k^{(1)}\right)S_k\left(P_k^{(1)} - K_k^{(1)}\right)\\&+ \sum_{j=1, j\neq k}^{N}\left(P_j^{(0)} - K_j^{(0)}\right)]x \qquad (25)\end{aligned}$$

which implies

$$x^T\left(P_k^{(1)} - P_k^{(0)}\right)S_k\left(P_k^{(1)} - P_k^{(0)}\right)x = 0, S_k \geq 0, \text{ for some } k \qquad (26)$$

or

$$S_k\left(P_k^{(1)} - P_k^{(0)}\right)x = 0, \text{ for some } k. \qquad (27)$$

Thus, we have obtained

$$\left(\mathbf{A_k} - S_k P_k^{(0)}\right)x = \left(\mathbf{A_k} - S_k P_k^{(1)}\right)x = \lambda x, x \neq 0, \text{ for some } k \qquad (28)$$

which is a contradiction due to the initial assumption that the matrices $\mathbf{A_k} - S_k P_k^{(0)}$ are stable for $\forall k$ so there is no such a k such that any of the matrices $\mathbf{A_k} - S_k P_k^{(1)}$ is unstable. We have already established from (21) that $P_k^{(1)} - K_k^{(1)} \geq 0, \forall k$. Using this fact in (22) we see that the right-hand side of this equation is negative semidefinite, that is, $-x^T M^{(2)} x \leq 0$. Repeating steps (23)–(27) for $i = 1$, it follows that $\mathbf{A_k} - S_k P_k^{(2)}$ are stable matrices for $\forall k$. Continuing the same procedure for $i = 2, 3, \cdots$, we conclude that

$$\begin{aligned}\mathbf{A_k} - S_k P_k^{(i)} \text{ stable } &\Rightarrow P_k^{(i+1)} - K_k^{(i+1)} \geq 0\\&\Rightarrow \mathbf{A_k} - S_k P_k^{(i+1)} \text{ stable } \forall i, \forall k. \qquad (29)\end{aligned}$$

Thus, the sequences $\left\{P_k^{(i)}\right\}$ are bounded from below by the sequences $\left\{K_k^{(i)}\right\}, \forall k,\ \forall i$.

Upper Bounds: In the following we establish that the sequences $\left\{P_k^{(i)}\right\}$ have the upper bounds, that is, the sequences $\left\{\overline{P_k^{(i)}}\right\} \geq \left\{P_k^{(i)}\right\}$ exist. In addition, these sequences (representing the upper bounds) monotonically converge from above to the required solutions of (6). Subtracting (6) from (9) we get

$$\begin{aligned}&\left(\mathbf{A_k} - S_k P_k^{(i)}\right)^T\left(P_k^{(i+1)} - P_k\right)\\&+ \left(P_k^{(i+1)} - P_k\right)\left(\mathbf{A_k} - S_k P_k^{(i)}\right) =\\&- \sum_{j=1, j\neq k}^{N} \pi_{kj}\left(P_j^{(i)} - P_j\right)\\&- \left(P_k^{(i)} - P_k\right)S_k\left(P_k^{(i)} - P_k\right),\ k = 1, 2, \cdots, N. \qquad (30)\end{aligned}$$

If for some iteration index i we have that

$$P_j^{(i)} - P_j \geq 0, \quad \text{for} \quad \forall j \qquad (31)$$

which can be obtained by choosing $P_j^{(0)} \geq P_j, \forall j$, then the right-hand side of (30) is negative semidefinite so that

$$P_k^{(i+1)} \geq P_k,\ \forall k = 1, 2, \cdots, N, \qquad \forall i = 1, 2, \cdots. \qquad (32)$$

Even more, the sequences of matrices $\left\{P_k^{(i+1)}\right\}$ obtained from the corresponding algebraic Lyapunov equations are monotonically convergent with P_k representing their limit points. To show this, first observe from (9)–(10) that the following holds

$$\begin{aligned}&\left(\mathbf{A_k} - S_k P_k^{(i+1)}\right)^T\left(P_k^{(i+2)} - P_k^{(i+1)}\right)\\&+ \left(P_k^{(i+2)} - P_k^{(i+1)}\right)\left(\mathbf{A_k} - S_k P_k^{(i+1)}\right) =\\&\sum_{j=1, j\neq k}^{N} \pi_{kj}\left(P_j^{(i)} - P_j^{(i+1)}\right)\\&+ \left(P_k^{(i)} - P_k^{(i+1)}\right)S_k\left(P_k^{(i)} - P_k^{(i+1)}\right)\\&k = 1, 2, \cdots, N;\ i = 1, 2, 3, \cdots. \qquad (33)\end{aligned}$$

If in addition we impose

$$P_j^{(0)} - P_j^{(1)} \geq 0, \quad \forall j \qquad (34)$$

then $P_k^{(i+2)} - P_k^{(i+1)} \leq 0, \forall i, \forall k$ so that monotonicity is obtained, that is

$$P_k^{(0)} \geq P_k^{(1)} \geq \cdots \geq P_k^{(i+1)} \geq P_k^{(i+2)} \geq \cdots P_k. \qquad (35)$$

Sequences (35) are convergent [6]–[9]. Thus, the crucial point is condition (34). Consider now the sequences of positive semidefinite matrices generated by the proposed algorithm (9)–(10) with the stabilizing matrices $P_k^{(0)}$ taken "arbitrarily large" such that condition (31) is satisfied. This can be always achieved by the stabilizability assumption. The required sequences are given by

$$\begin{aligned}&\left(\mathbf{A_k} - S_k \overline{P_k^{(i)}}\right)^T \overline{P_k^{(i+1)}} + \overline{P_k^{(i+1)}}\left(\mathbf{A_k} - S_k \overline{P_k^{(i)}}\right)\\&= -\overline{P_k^{(i)}} S_k \overline{P_k^{(i)}} - \overline{Q_k^{(i)}}, \overline{P_k^{(0)}} \geq P_k,\ k = 1, 2, \cdots, N \qquad (36)\end{aligned}$$

where

$$\overline{Q_k^{(i)}} = Q_k + \sum_{j=1, j\neq k}^{N} \pi_{kj}\overline{P_j^{(i)}} \geq 0. \qquad (37)$$

Note that the sequences $\left\{\overline{P_k^{(i)}}\right\}$ and $\left\{P_k^{(i)}\right\}$ are obtained from the same algorithm (9)–(10), and they only differ in the initial points

$P_k^{(0)} \leq \overline{P_k^{(0)}}$. It can be easily shown by subtracting (36)–(37) from (9)–(10) that for any $0 \leq P_k^{(0)} \leq \overline{P_k^{(0)}}$ the sequences $\left\{P_k^{(i)}\right\}$ are dominated by the sequences $\left\{\overline{P_k^{(i)}}\right\}$, that is, the latter sequences represent the upper bounds for the former ones so that the corresponding inequalities (11) are satisfied. The sequences obtained from (36)–(37) must also satisfy condition (34). For $i = 0$, we get from (36)

$$\begin{aligned}&\left(\mathbf{A_k} - S_k\overline{P_k^{(0)}}\right)^T\left(\overline{P_k^{(1)}} - \overline{P_k^{(0)}}\right)\\&+\left(\overline{P_k^{(1)}} - \overline{P_k^{(0)}}\right)\left(\mathbf{A_k} - S_k\overline{P_k^{(0)}}\right)\\&=\overline{P_k^{(0)}}S_k\overline{P_k^{(0)}} - \overline{P_k^{(0)}}\mathbf{A_k}\\&-\mathbf{A_k^T}\overline{P_k^{(0)}} - Q_k^{(0)},\\&k = 1, 2, \cdots, N.\end{aligned} \tag{38}$$

Condition (34) will be satisfied if the right-hand side of (38) is positive semidefinite. It is shown in the Appendix in Lemma 1 that the right-hand side of (38) is positive semidefinite for $\overline{P_k^{(0)}} \geq P_k$, so that both conditions (31) and (34) are satisfied. Of course the algorithm (36)–(37) will converge to the desired solutions of (6)–(7), but very large values of $\overline{P_k^{(0)}}$ might slow the convergence process. Thus, the sequences obtained from (36) have only theoretical importance to establish the upper bounds for the sequences $\left\{P_k^{(i)}\right\}$ since $0 \leq P_k^{(0)} \leq \overline{P_k^{(0)}}, \forall k$ imply $0 \leq P_k^{(i)} \leq \overline{P_k^{(i)}}, \forall i, \forall k$. We use the sequences $\left\{P_k^{(i)}\right\}$ for the actual computations. Note that the sequences $\left\{P_k^{(i)}\right\}$ are nested between two sets of sequences

$$\left\{K_k^{(i)}\right\} \leq \left\{P_k^{(i)}\right\} \leq \left\{\overline{P_k^{(i)}}\right\}, \ \forall i, \ \forall k.$$

Since both sequences $\left\{K_k^{(i)}\right\}$ and $\left\{\overline{P_k^{(i)}}\right\}$ converge to the required solutions of (6) so do the sequences $\left\{P_k^{(i)}\right\}, \forall k$. Initial conditions for $\left\{P_k^{(i)}\right\}, \forall k$ can be chosen as arbitrary positive semidefinite stabilizing matrices, $P_k^{(0)} \geq 0$.

IV. Numerical Example

Example 1: The following example was considered in [2]

$$\begin{aligned}&A_1 = diag(-2.5, -3, -2), \qquad B_1 = diag\left(\sqrt{0.5}, 1, 1\right),\\&\qquad Q_1 = diag(25, 1, 11)\\&A_2 = diag(-2.5, 5, 5), \qquad B_2 = diag\left(\sqrt{0.5}, 1, \sqrt{0.5}\right),\\&\qquad Q_2 = diag(37.5, 704, 34.5)\\&A_1 = diag(2, -3, -2), \qquad B_3 = diag\left(\sqrt{0.5}, 1, 1\right),\\&\qquad Q_3 = diag(10, 16, 21)\\&R_1 = R_2 = R_3 = I_3\\&p_{11} = -3, \ p_{12} = 0.5, \ p_{13} = 2.5,\\&\qquad p_{22} = p_{33} = p_{21} = p_{23} = p_{31} = p_{32} = 0.\end{aligned}$$

Using the initial conditions obtained from the decoupled Riccati algebraic equations [like in (15)] it took only five iterations for the proposed Lyapunov iterations algorithm (9)–(10) to achieve the accuracy of $O(10^{-15})$. Taking the initial guesses as $P_k^{(0)} = 100I_3, k = 1, 2, 3$ we got the accuracy of $O(10^{-15})$ after 10 iterations.

TABLE I
Error Propagation

Iteration	Error
1	9.6000×10^{-2}
3	3.3579×10^{-4}
5	3.2379×10^{-6}
7	3.5113×10^{-8}
10	4.2811×10^{-11}
12	4.9626×10^{-13}
14	4.5259×10^{-15}

Example 2: Consider the following fourth-order jump linear control problem

$$A_1 = \begin{bmatrix}-2.1051 & -1.1648 & 0.9347 & 0.5194\\-0.0807 & -2.8949 & 0.3835 & 0.8310\\0.6914 & 10.5940 & -36.8199 & 3.8560\\1.0692 & 13.4230 & 22.1185 & -13.1801\end{bmatrix},$$

$$A_2 = \begin{bmatrix}-2.6430 & -1.2497 & 0.5269 & 0.6539\\-0.7910 & -2.8570 & 0.0920 & 0.4160\\21.0357 & 22.8659 & -26.4655 & -1.7214\\27.3096 & 7.8736 & -3.8604 & -29.5345\end{bmatrix}$$

$$B_1 = \begin{bmatrix}0.7564\\0.9910\\9.8255\\7.2266\end{bmatrix}, \qquad B_2 = \begin{bmatrix}0.3653\\0.2470\\7.5336\\6.5152\end{bmatrix},$$

$$Q_1 = Q_2 = \begin{bmatrix}1 & 0 & 1 & 0\\0 & 0 & 0 & 0\\1 & 0 & 1 & 0\\0 & 0 & 0 & 0\end{bmatrix},$$

$$\Pi = \begin{bmatrix}-2 & 2\\1.5 & -1.5\end{bmatrix}, \qquad R_1 = R_2 = 1.$$

The following solutions have been obtained with the accuracy of $O(10^{-15})$ after 14 iterations

$$P_1 = \begin{bmatrix}0.2408 & 0.0705 & 0.0393 & 0.0182\\0.0705 & 0.0308 & 0.0085 & 0.0064\\0.0393 & 0.0085 & 0.0157 & 0.0025\\0.0182 & 0.0064 & 0.0025 & 0.0016\end{bmatrix},$$

$$P_2 = \begin{bmatrix}0.5026 & 0.1343 & 0.0518 & 0.0097\\0.1343 & 0.0485 & 0.0138 & 0.0026\\0.0518 & 0.0138 & 0.0193 & 0.0002\\0.0097 & 0.0026 & 0.0002 & 0.0003\end{bmatrix}.$$

The initial conditions for this problem were obtained by using solutions of the decoupled algebraic Riccati equations. Error propagation is given in Table I, where the error is defined as

$$max\left(\left\|\Re_1\left(P_1^{(i)}, P_2^{(i)}\right)\right\|_2, \left\|\Re_2\left(P_1^{(i)}, P_2^{(i)}\right)\right\|_2\right).$$

All simulation results in this paper are obtained by using MATLAB.

V. Conclusion

Assuming that the positive semidefinite stabilizing solutions, P_k, of the coupled algebraic Riccati equations of jump linear systems exist, they can be found in terms of decoupled algebraic Lyapunov equations by using the proposed algorithm (9)–(10).

APPENDIX

Lemma 1: There exist $P_k^{(0)} \geq P_k, k = 1, 2, \cdots, N$, where P_k are unique positive semidefinite stabilizing solutions of (6)–(7), such that

$$\Re\left(P_k^{(0)}\right) = P_k^{(0)} S_k P_k^{(0)} - \mathbf{A}_\mathbf{k}^\mathbf{T} P_k^{(0)} - P_k^{(0)} \mathbf{A}_\mathbf{k} - Q_k - \sum_{j=1, j\neq k}^{N} \pi_{kj} P_j^{(0)} \geq 0, \quad k = 1, 2, \cdots, N. \tag{39}$$

Proof: Let $P_k^{(0)} = P_k + X_k$ with $X_k \geq 0$, then

$$\Re\left(P_k^{(0)}\right) = \Re(P_k) + P_k S_k X_k + X_k S_k P_k + X_k S_k X_k - A_k^T X_k - X_k A_k - \pi_{kk} X_k - \sum_{j=1, j\neq k}^{N} \pi_{kj} X_j. \tag{40}$$

Using the fact that $\Re(P_k) = 0$ we rewrite (40) as

$$\Re\left(P_k^{(0)}\right) + \sum_{j=1}^{N} \pi_{kj} X_j - X_k S_k X_k = -(A_k - S_k P_k)^T X_k - X_k (A_k - S_k P_k). \tag{41}$$

Since $X_k \geq 0$ and $A_k - S_k P_k$ are stable matrices $\forall k$ by Assumption 2, it follows that the left-hand side of the Lyapunov equations (41) must be positive semidefinite, that is

$$\Re\left(P_k^{(0)}\right) + \sum_{j=1}^{N} \pi_{kj} X_j - X_k S_k X_k \geq 0. \tag{42}$$

Choosing $X_1 = X_2 = \cdots = X_k = \cdots = X_N$ and using the fact that $\pi_{kk} = -\sum_{j=1, j\neq k}^{N} \pi_{kj}$, it follows that

$$\Re\left(P_k^{(0)}\right) \geq X_k S_k X_k \geq 0 \tag{43}$$

which completes the proof of Lemma 1.

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Fast Time-Varying Phasor Analysis in the Balanced Three-Phase Large Electric Power System

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***Abstract*—Traditional phasor representation of sinusoidal signals, the standard analytical tool for power system stability analysis, is limited by the quasistationary assumption on the speeds of the phasor states. This paper provides a rigorous formulation of a time-varying phasor representation for the balanced three-phase large power system with no restrictions on the speeds. Power balance equations become a set of differential equations in the phasor dynamic states and singularly perturbed behavior of the resulting dynamics is explored.**

I. INTRODUCTION

Phasors were introduced around the turn of the century to facilitate computations and analysis of power systems in stationary operation. It is a mathematical transformation, which eliminates the 60 Hz "carrier," the only time-varying element in the stationary case. This can be viewed as a form of demodulation. In the 1920's it was found that the assumption of stationarity can be relaxed to an approximation, the "quasistationary" assumption, which allows voltages, currents, or power to vary "slowly." By fortunate coincidence, angular rotor swings in the power system (which are typically slower than 1/2 or 1/3 Hz) provide a quasistationary environment found experimentally to have quite satisfactory accuracy for computation of "transient stability" (limits have been mathematically analyzed recently [1], [2]). Such transient swings were the basic concern, the principal problem area, of fast power system dynamics until about the 1970's. More recently faster phenomena connected with voltage stability are often outside the "quasistationary range," a fact which leads to questionable results when ignored. It is then necessary to properly define a "time-varying phasor" concept as a mathematical transformation and establish its properties such as the phasor calculus and the validity of the phasor power-flow equations (reactive power, real power, etc.). This is done in this note. It needs to be emphasized that the conventional π circuit representation used in this note itself imposes validity limitations (although at a higher speed level) on time variations. The ultimate, precise distributed parameter phasors have now been introduced, and they make it possible to evaluate and upgrade the equivalent π or RLC approximation [3].

Earlier approaches (e.g., [4]) use physical reasoning and intuitive deduction to try to reach a general time-dependent phasor concept. Under the assumption that the system (including the transmission

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