Exact Decomposition of the Algebraic Riccati Equation of Deterministic Multimodeling Optimal Control Problems

Cyril Coumarbatch and Zoran Gajic

***Abstract*—In this paper we show how to exactly decompose the algebraic Riccati equations of deterministic multimodeling in terms of one pure-slow and two pure-fast algebraic Riccati equations. The algebraic Riccati equations obtained are of reduced-order and nonsymmetric. However, their $O(\epsilon)$ perturbations (where $\epsilon = \|{}^{\epsilon_1}_{\epsilon_2}\|$ and ϵ_1, ϵ_2 are small positive singular perturbation parameters) are symmetric. The Newton method is perfectly suited for solving the nonsymmetric reduced-order pure-slow and pure-fast algebraic Riccati equations since excellent initial guesses are available from their $O(\epsilon)$ perturbed reduced-order symmetric algebraic Riccati equations that can be solved rather easily. The proposed decomposition scheme might facilitates new approaches to mutimodeling control problems that are conceptually simpler and numerically more efficient than the ones previously used.**

***Index Terms*—Algebraic Riccati equation, linear systems, multimodeling, optimal control, singular perturbations.**

I. Introduction

The concept of multimodeling was introduced to the control audience in [1]. Since then, the deterministic and stochastic multimodeling control and filtering problems have been studied by several researchers [2]–[13]. The multimodeling problems arise in large scale dynamic systems that have multiple decision makers and multiple information channels (structures). Large scale systems are composed of several subsystems and characterized by the presence of slow and fast dynamics and weak and strong interconnections among state variables. It is known from [1] that theory of singular perturbations is very well suited to capture the multimodeling structure of interconnected large scale systems displaying slow and fast dynamics.

The optimal solution to deterministic linear-quadratic optimal multimodel control problems requires the solution of the multiparameter singularly perturbed algebraic Riccati equation. In this paper we show how to exactly decompose the corresponding algebraic Riccati equation in terms of independent one pure-slow and two pure-fast, reduced-order, algebraic Riccati equations.

The results obtained in this paper represent very powerful tools for simplified derivations of the optimal multimodel control and strategies. In that respect, the results of [2], [11] (see also [14], and [13] can be obtained with perfect accuracy with only minor modifications. The extension to the Pareto multimodeling strategies of [1], [9] will require a generalization of the results presented in this paper to the Pareto game algebraic Riccati equation. The extension to the multimodeling team problems [7] will require much more work along the lines considered in this paper. The Nash multimodeling strategies of [3] and [5] can be similarly studied under the assumption that the results of this paper can be applied to the coupled Nash algebraic Riccati equations.

By using the results of this paper, the multimodeling strategies could be implemented with perfect accuracy. It is known that all multimodeling results presently available in the literature are of the accuracy of $O(\epsilon)$.[1] Several examples done in [15]–[18] indicate that an $O(\epsilon)$ accuracy is very often not sufficient. Hence, the development of the more accurate techniques for singularly perturbed control systems is mandatory.

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C. Coumarbatch is with the Department of Mathematics, Rutgers University, New Brunswick, NJ 08903 USA (e-mail: cec@math.rutgers.edu).

Z. Gajic is with the Department of Electrical and Computer Engineering, Rutgers University, Piscataway, NJ 08854-8058 USA (e-mail: gajic@ece.rutgers.edu).

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II. Decomposition of the Regulator Algebraic Riccati Equation

The multimodeling structure is defined by a linear dynamic system that has one slow and two fast subsystems. The fast subsystems are strongly connected to the slow subsystem and weakly connected (or not connected) among themselves. Such large scale systems describe dynamics of several real physical systems, for example, power systems [1] and automobiles [13], [19]. The corresponding multimodeling representation of [1] is defined by

$$\begin{aligned}\dot{x}_0(t) &= A_{00}x_0(t) + A_{01}x_1(t) + A_{02}x_2(t)\\ &\quad + B_{01}u_1(t) + B_{02}u_2(t)\\ \epsilon_1\dot{x}_1(t) &= A_{10}x_0(t) + A_{11}x_1(t) + \epsilon_3 A_{12}x_2(t)\\ &\quad + B_{11}u_1(t) + \epsilon_3 B_{12}u_2(t)\\ \epsilon_2\dot{x}_2(t) &= A_{20}x_0(t) + \epsilon_3 A_{21}x_1(t) + A_{22}x_2(t)\\ &\quad + \epsilon_3 B_{21}u_1(t) + B_{22}u_2(t)\end{aligned} \tag{1}$$

where

$x_0 \in \Re^{n_0}$	slow state variables;
$x_1 \in \Re^{n_1},\ x_2 \in \Re^{n_2}$	fast state variables;
$u_1 \in \Re^{m_1},\ u_2 \in \Re^{m_2}$	control inputs.

ϵ_3 is a small weak coupling parameter, and ϵ_1 and ϵ_2 are small positive singular perturbation parameters of the same order of magnitude, that is $0 < k_1 \le (\epsilon_2/\epsilon_1) \le k_2 < \infty$. In order to simplify derivations, without loss of generality, we assume that the fast state variables are not connected among themselves, that is, we set the weak coupling parameter ϵ_3 to zero.

In the deterministic optimal control of the above multimodeling structure, the quadratic performance criterion has to be minimized by the proper choice of the control variables $u_1(t)$ and $u_2(t)$. The performance criterion is given by

$$\begin{aligned}J &= \frac{1}{2}\int_0^{+\infty}\left[x^T(t)Qx(t) + u^T(t)Ru(t)\right]dt\\ Q &= Q^T \ge 0,\ R = R^T > 0\end{aligned} \tag{2}$$

where

$$\begin{aligned}x(t) &= \begin{bmatrix} x_0(t)\\ x_1(t)\\ x_2(t)\end{bmatrix},\quad u(t) = \begin{bmatrix} u_1(t)\\ u_2(t)\end{bmatrix},\quad R = \begin{bmatrix} R_1 & 0\\ 0 & R_2\end{bmatrix}\\ Q &= \begin{bmatrix} Q_{00} & Q_{01} & Q_{02}\\ Q_{01}^T & Q_{11} & 0\\ Q_{02}^T & 0 & Q_{22}\end{bmatrix}\\ Q &= q^T q = \begin{bmatrix} q_{01} & q_{11} & 0\\ q_{02} & 0 & q_{22}\end{bmatrix}^T \begin{bmatrix} q_{01} & q_{11} & 0\\ q_{02} & 0 & q_{22}\end{bmatrix}\\ &= \begin{bmatrix} q_{01}^T q_{01} + q_{02}^T q_{02} & q_{01}^T q_{11} & q_{02}^T q_{22}\\ q_{11}^T q_{01} & q_{11}^T q_{11} & 0\\ q_{22}^T q_{02} & 0 & q_{22}^T q_{22}\end{bmatrix}.\end{aligned} \tag{3}$$

In the general multimodeling case, all zero-elements in matrices R and Q can be replaced by $O(\epsilon_3)$ elements.

[1] $O(\epsilon^i)$ is defined by $O(\epsilon^i) < c\epsilon^i$, where c is a bounded constant and i is a real number. In this paper $\epsilon = \|{}^{\epsilon_1}_{\epsilon_2}\|$.

In the multimodeling problem one proceeds with constructing two different models of (1), obtained by setting $\epsilon_1 = 0$, which leads to the first model for the first controller, and by setting $\epsilon_2 = 0$, which produces the second model for the second controller. The rationale for this is the fact that each controller "sees" the slow dynamics of both subsystems and only its own fast dynamics. Thus, the fast dynamics of the other subsystem is approximated by an algebraic equation (the corresponding ϵ_i is set to zero). The same approximation is done for the performance criterion (2), hence two performance criteria are obtained, which leads to multicriteria optimization problem. Depending on the actual problem setup, very often described by differential games, the two controllers find their own optimal strategies and apply such strategies to the global system defined by (1). In such a way obtained, the multimodeling strategy is well posed if the performance criterion under the multimodeling strategy is $O(\epsilon)$ close to the global optimal control strategy obtained by performing direct optimization on the original system and the original performance criterion.

In this paper, we propose a method for the exact decomposition of the optimal control associated with (1) and (2) such that the optimal solution is obtained in terms of three independent, reduced-order algebraic Riccati equations, representing one slow and two fast subsystems. The idea presented in this paper will allow the development of new techniques for new setups and more efficient solutions of the corresponding multimodeling problems.

The optimal feedback solution to (1), (2) is given by

$$u_{opt}(t) = -R^{-1}B^T P x(t) \tag{4}$$

where P is the positive semidefinite stabilizing solution of the algebraic Riccati equation

$$A^T P + PA + Q - PSP = 0, \qquad S = BR^{-1}B^T \tag{5}$$

with

$$A = \begin{bmatrix} A_{00} & A_{01} & A_{02} \\ \frac{1}{\epsilon_1}A_{10} & \frac{1}{\epsilon_1}A_{11} & 0 \\ \frac{1}{\epsilon_2}A_{20} & 0 & \frac{1}{\epsilon_2}A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{01} & B_{02} \\ \frac{1}{\epsilon_1}B_{11} & 0 \\ 0 & \frac{1}{\epsilon_2}B_{22} \end{bmatrix}$$

$$P = \begin{bmatrix} P_{00} & \epsilon_1 P_{01} & \epsilon_2 P_{02} \\ \epsilon_1 P_{01}^T & \epsilon_1 P_{11} & \sqrt{\epsilon_1\epsilon_2}P_{12} \\ \epsilon_2 P_{02}^T & \sqrt{\epsilon_1\epsilon_2}P_{12}^T & \epsilon_2 P_{22} \end{bmatrix}$$

$$S = \begin{bmatrix} S_{00} & \frac{1}{\epsilon_1}S_{01} & \frac{1}{\epsilon_2}S_{02} \\ \frac{1}{\epsilon_1}S_{01}^T & \frac{1}{\epsilon_1^2}S_{11} & 0 \\ \frac{1}{\epsilon_2}S_{02}^T & 0 & \frac{1}{\epsilon_2^2}S_{22} \end{bmatrix}$$

$$\begin{aligned} S_{00} &= B_{01}R^{-1}B_{01}^T + B_{02}R^{-1}B_{02}^T, \quad S_{01} = B_{01}R^{-1}B_{11}^T \\ S_{02} &= B_{02}R^{-1}B_{22}^T, \quad S_{ii} = B_{ii}R^{-1}B_{ii}^T, \qquad i = 1, 2. \end{aligned} \tag{6}$$

The scaling of the matrix P is done according to nature of the solution of (5) as discussed in [9], [10]. The required solution of the algebraic Riccati (5) exists under the standard assumption [20].

Assumption 1: The triple (A, B, q) is stabilizable-detectable. Note that the multimodeling optimal control problem is studied under the following assumption [1].

Assumption 2: The triples (A_s, B_s, q_s) and (A_{ii}, B_{ii}, q_{ii}), $i = 1, 2$, are stabilizable-detectable.

The matrices A_s, B_s, q_s are given by [1], [10]

$$\begin{aligned} A_s &= A_{00} - A_{01}A_{11}^{-1}A_{10} - A_{02}A_{22}^{-1}A_{20} \\ B_s &= [\,B_{1s} \quad B_{2s}\,], \quad B_{is} = B_{0i} - A_{0i}A_{ii}^{-1}B_{io}, \\ &\qquad i = 1, 2 \\ Q_s &= q_s^T q_s = q_{1s}^T q_{1s} + q_{2s}^T q_{2s}, \quad q_{is} = q_{0i} - q_{ii}A_{ii}^{-1}A_{i0}, \\ &\qquad i = 1, 2. \end{aligned}$$

In this paper, the A_s, B_s, Q_s matrices will be redefined later on as a part of the proposed design methodology. Note that for sufficiently small values of $\epsilon = \|{}^{\epsilon_1}_{\epsilon_2}\|$, Assumption 2 is equivalent to Assumption 1 [1].

The derivations that follow will require Assumption 2. Consider the Hamiltonian matrix corresponding to (1) and (2)

$$\begin{bmatrix} \dot{x}(t) \\ \dot{p}(t) \end{bmatrix} = \boldsymbol{H} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} = \begin{bmatrix} A & -S \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} \tag{7}$$

where $p(t)$ represents the so-called co-state system variables compatibly partitioned as $p^T(t) = [\,p_0^T(t) \quad \epsilon_1 p_1^T(t) \quad \epsilon_2 p_2^T(t)\,]$. Let E_1 be the permutation matrix defined by

$$E_1 = \begin{bmatrix} I_{n_0} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n_0} & 0 & 0 \\ 0 & I_{n_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\epsilon_1}I_{n_1} & 0 \\ 0 & 0 & I_{n_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\epsilon_2}I_{n_2} \end{bmatrix}. \tag{8}$$

The similarity transformation E_1 applied to (7) produces

$$\begin{aligned} \begin{bmatrix} \dot{x}_0 \\ \dot{p}_0 \\ \dot{x}_1 \\ \dot{p}_1 \\ \dot{x}_2 \\ \dot{p}_2 \end{bmatrix} &= E_1 \begin{bmatrix} A & -S \\ -Q & -A^T \end{bmatrix} E_1^{-1} \begin{bmatrix} x_0 \\ p_0 \\ x_1 \\ p_1 \\ x_2 \\ p_2 \end{bmatrix} \\ &= \begin{bmatrix} T_{00} & T_{01} & T_{02} \\ \frac{1}{\epsilon_1}T_{10} & \frac{1}{\epsilon_1}T_{11} & 0 \\ \frac{1}{\epsilon_2}T_{20} & 0 & \frac{1}{\epsilon_2}T_{22} \end{bmatrix} \begin{bmatrix} x_0 \\ p_0 \\ x_1 \\ p_1 \\ x_2 \\ p_2 \end{bmatrix} = T \begin{bmatrix} x_0 \\ p_0 \\ x_1 \\ p_1 \\ x_2 \\ p_2 \end{bmatrix} \end{aligned} \tag{9}$$

where

$$\begin{aligned} T_{00} &= \begin{bmatrix} A_{00} & -S_{00} \\ -Q_{00} & -A_{00}^T \end{bmatrix}, \quad T_{01} = \begin{bmatrix} A_{01} & -S_{01} \\ -Q_{01} & -A_{10}^T \end{bmatrix} \\ T_{02} &= \begin{bmatrix} A_{02} & -S_{02} \\ -Q_{02} & -A_{20}^T \end{bmatrix}, \quad T_{10} = \begin{bmatrix} A_{10} & -S_{01}^T \\ -Q_{01}^T & -A_{01}^T \end{bmatrix} \\ T_{11} &= \begin{bmatrix} A_{11} & -S_{11} \\ -Q_{11} & -A_{11}^T \end{bmatrix}, \quad T_{12} = 0, \quad T_{20} = \begin{bmatrix} A_{20} & -S_{02}^T \\ -Q_{02}^T & -A_{02}^T \end{bmatrix} \\ T_{22} &= \begin{bmatrix} A_{22} & -S_{22} \\ -Q_{22} & -A_{22}^T \end{bmatrix}, \quad T_{21} = 0. \end{aligned} \tag{10}$$

Note that the above transformation combines in pairs the slow state/co-state and fast state/co-state variables such that (9) has the singularly perturbed structure. It should be pointed out that due to the second part of Assumption 2, the fast Hamiltonian matrices T_{11} and T_{22} are nonsingular [20]. In addition, the first part of Assumption 2 implies that the slow Hamiltonian matrix given by

$$T_s = \begin{bmatrix} A_s & -S_s \\ -Q_s & -A_s^T \end{bmatrix} = T_{00} - T_{01}T_{11}^{-1}T_{10} - T_{02}T_{22}^{-1}T_{20} \tag{11}$$

is nonsingular. Note that T_s is obtained from (9) by extracting the slow subsystem, that is, by multiplying the fast derivatives, respectively, by ϵ_1 and ϵ_2 and setting them to zero. This expression also gives new definitions for matrices A_s, Q_s, S_s, with $S_s = B_s R_s^{-1} B_s^T$. The procedure for obtaining independently R_s can be found in [1]. For the purpose of this paper we need only S_s. Due to the fact that R_s is invertible, it follows that stabilizability of (A_s, B_s) is equivalent to stabilizability of $(A_s, \sqrt{S_s})$.

The singularly perturbed system defined in (9) can be block-diagonalized by using the transformation derived in [21] as shown in (12) at the bottom of the page. The corresponding inverse transformation is shown in (13) at the bottom of the page.

In the above transformation the matrices $H_j, L_j, j = 1, 2, 3$, satisfy

$$\begin{aligned}
0 &= T_{11}L_1 - T_{10} - \epsilon_1 L_1(T_{00} - T_{01}L_1 - T_{02}L_2 + T_{02}L_3L_1)\\
0 &= T_{22}L_2 - \alpha L_3 T_{10} - T_{20} - \epsilon_2 L_2(T_{00} - T_{02}L_2)\\
0 &= T_{22}L_3 - \alpha L_3 T_{11} - \epsilon_2 L_2(T_{01} - T_{02}L_3)\\
0 &= -H_1T_{11} - \epsilon_1 H_1 L_1(T_{01} - T_{02}L_3) + (T_{01} - T_{02}L_3)\\
&\quad + \epsilon_1(T_{00} - T_{01}L_1 - T_{02}L_2 + T_{02}L_3L_1)H_1\\
0 &= -H_2T_{22} + \alpha T_{11}H_2 + \epsilon_2 L_1(T_{01} - T_{02}L_3)H_2\\
&\quad + (L_1 - \epsilon_2 H_2 L_2)T_{02}\\
0 &= -H_3T_{22} - \epsilon_2 H_3 L_2 T_{02} - \epsilon_2(T_{01} - T_{02}L_3)H_2 - T_{02}\\
&\quad + \epsilon_2(T_{00} - T_{01}L_1 - T_{02}L_2 + T_{02}L_3L_1)H_3\\
&\qquad 0 < k_1 \le \alpha = \frac{\epsilon_2}{\epsilon_1} \le k_2 < \infty. \qquad (14)
\end{aligned}$$

Even though the above algebraic equations are nonlinear, it can be noticed that all nonlinear terms are multiplied by the small singular perturbation parameters. Hence, an $O(\epsilon)$ perturbation of (14) produces a *set of decoupled linear algebraic equations*. Solutions of this set of linear algebraic equations represent excellent initial conditions for the fixed point algorithm to be used for solving (14) since $L_j = L_j^{(0)} + O(\epsilon)$, $H_j = H_j^{(0)} + O(\epsilon), j = 1, 2, 3$. An $O(\epsilon)$ perturbation of (14) is given by

$$\begin{aligned}
0 &= T_{11}L_1^{(0)} - T_{10} \Rightarrow L_1^{(0)} = T_{11}^{-1}T_{10}\\
0 &= T_{22}L_2^{(0)} - \alpha L_3^{(0)}T_{10} - T_{20} \Rightarrow L_2^{(0)} = T_{22}^{-1}T_{20}\\
0 &= T_{22}L_3^{(0)} - \alpha L_3^{(0)}T_{11} \Rightarrow L_3^{(0)} = 0\\
0 &= -H_1^{(0)}T_{11} + (T_{01} - T_{02}L_3^{(0)}) \Rightarrow H_1^{(0)} = T_{01}T_{11}^{-1}\\
0 &= -H_2^{(0)}T_{22} + \alpha T_{11}H_2^{(0)} + L_1^{(0)}T_{02}\\
0 &= -H_3^{(0)}T_{22} - T_{02} \Rightarrow H_3^{(0)} = -T_{02}T_{22}^{-1}. \qquad (15)
\end{aligned}$$

It can be seen that these linear algebraic equations can be solved rather easily due to their decoupled structure and the fact that the Hamiltonian matrices T_{11} and T_{22} are nonsingular, which is the consequence of Assumption 2. Note that the equations for $L_3^{(0)}$ and $H_2^{(0)}$ are the Sylvester linear algebraic equations. The unique solutions of these equation exist under the following assumption [22].

Assumption 3: The Hamiltonian matrices T_{22} and αT_{11} have no eigenvalues in common.

Due to the above assumption, the existence of $L_1^{(0)}, L_2^{(0)}, L_3^{(0)}$ is not uniform with respect to α. Since the unique solutions for $L_1^{(0)}, L_2^{(0)}, L_3^{(0)}$ exist under Assumption 3, then by the Implicit Function Theorem [23], the unique solutions L_1, L_2, L_3 exist for sufficiently small values of ϵ. The fixed point algorithm for solving (14) is given by

$$\begin{aligned}
&T_{11}L_1^{(i+1)}\\
&\quad = T_{10} + \epsilon_1 L_1^{(i)}\left(T_{00} - T_{01}L_1^{(i)} - T_{02}L_2^{(i)} + T_{02}L_3^{(i)}L_1^{(i)}\right)\\
&T_{22}L_2^{(i+1)} - \alpha L_3^{(i+1)}T_{10}\\
&\quad = T_{20} + \epsilon_2 L_2^{(i)}\left(T_{00} - T_{02}L_2^{(i)}\right)\\
&T_{22}L_3^{(i+1)} - \alpha L_3^{(i+1)}T_{11}\\
&\quad = \epsilon_2 L_2^{(i)}\left(T_{01} - T_{02}L_3^{(i)}\right)\\
&H_1^{(i+1)}T_{11}\\
&\quad = -\epsilon_1 H_1^{(i)}L_1^{(i)}\left(T_{01} - T_{02}L_3^{(i)}\right) + \left(T_{01} - T_{02}L_3^{(i)}\right)\\
&\qquad + \epsilon_1\left(T_{00} - T_{01}L_1^{(i)} - T_{02}L_2^{(i)} + T_{02}L_3^{(i)}L_1^{(i)}\right)H_1^{(i)}\\
&H_2^{(i+1)}T_{22} - \alpha T_{11}H_2^{(i+1)} - L_1^{(i+1)}T_{02}\\
&\quad = \epsilon_2 L_1^{(i)}\left(T_{01} - T_{02}L_3^{(i)}\right)H_2^{(i)} - \epsilon_2 H_2^{(i)}L_2^{(i)}T_{02}\\
&H_3^{(i+1)}T_{22}\\
&\quad = -\epsilon_2 H_3^{(i)}L_2^{(i)}T_{02} - \epsilon_2\left(T_{01} - T_{02}L_3^{(i)}\right)H_2^{(i)} - T_{02}\\
&\qquad + \epsilon_2\left(T_{00} - T_{01}L_1^{(i)} - T_{02}L_2^{(i)} + T_{02}L_3^{(i)}L_1^{(i)}\right)H_3^{(i)}. \qquad (16)
\end{aligned}$$

Theorem 1: Under Assumptions 2 and 3, the fixed point algorithm (16) converges to the solutions of (15) with the rate of convergence of $O(\epsilon)$, that is

$$\begin{aligned}
\left\|L_j^{(i+1)} - L_j^{(i)}\right\| &= O(\epsilon), \qquad j = 1, 2, 3;\ i = 0, 1, 2, \cdots\\
\left\|H_j^{(i+1)} - H_j^{(i)}\right\| &= O(\epsilon), \qquad j = 1, 2, 3;\ i = 0, 1, 2, \cdots
\end{aligned} \qquad (17)$$

$$\begin{aligned}
\left\|L_j - L_j^{(i)}\right\| &= O(\epsilon^{i+1}), \qquad j = 1, 2, 3;\ i = 0, 1, 2, \cdots\\
\left\|H_j - H_j^{(i)}\right\| &= O(\epsilon^{i+1}), \qquad j = 1, 2, 3;\ i = 0, 1, 2, \cdots.
\end{aligned} \qquad (18)$$

$$K = \begin{bmatrix} I_{n_0} - \epsilon_1 H_1 L_1 + \epsilon_1\epsilon_2 H_1 H_2 L_2 + \epsilon_2 H_3 L_2 & -\epsilon_1 H_1 + \epsilon_1\epsilon_2 H_1 H_2 L_3 + \epsilon_2 H_3 L_2 & \epsilon_2(H_3 + \epsilon_1 H_1 H_2)\\ L_1 - \epsilon_2 H_2 L_2 & I_{n_1} - \epsilon_2 H_2 L_3 & -\epsilon_2 H_2\\ L_2 & L_3 & I_{n_2} \end{bmatrix} \qquad (12)$$

$$K^{-1} = \begin{bmatrix} I_{n_0} & \epsilon_1 H_1 & -\epsilon_2 H_3\\ -L_1 & I_{n_1} - \epsilon_1 H_1 L_1 & \epsilon_2(H_2 + H_3 L_1)\\ -L_2 + L_1 L_3 & \epsilon_1 H_1(L_1 L_3 - L_2) - L_3 & I_{n_2} + \epsilon_2(H_3 L_3 - H_2 L_3 - H_3 L_3 L_1) \end{bmatrix} \qquad (13)$$

Proof: The proof of this theorem is rather lengthy. It can be found in [24].

An algorithm for solving the L-equations by using the Newton method, with the solutions of (15) playing the role of the initial conditions, is also developed in [24].

By applying the transformation K to (9), the system is transformed into the new coordinates with completely decoupled slow and fast dynamics

$$\begin{bmatrix} \dot{\eta}_{01}(t) \\ \dot{\eta}_{02}(t) \\ \epsilon_1 \dot{\eta}_{11}(t) \\ \epsilon_1 \dot{\eta}_{12}(t) \\ \epsilon_2 \dot{\eta}_{21}(t) \\ \epsilon_2 \dot{\eta}_{22}(t) \end{bmatrix} = \begin{bmatrix} D_0 & 0 & 0 \\ 0 & D_1 & 0 \\ 0 & 0 & D_2 \end{bmatrix} \begin{bmatrix} \eta_{01}(t) \\ \eta_{02}(t) \\ \eta_{11}(t) \\ \eta_{12}(t) \\ \eta_{21}(t) \\ \eta_{22}(t) \end{bmatrix} \tag{19}$$

with

$$\begin{aligned} D_0 &= T_{00} - T_{01}L_1 - T_{02}L_2 + T_{02}L_3L_1 \triangleq \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \\ D_1 &= T_{11} + \epsilon_1 L_1(T_{01} - T_{02}L_3) \triangleq \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \\ D_2 &= T_{22} + \epsilon_2 L_2 T_{02} \triangleq \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}. \end{aligned} \tag{20}$$

In (19) $\eta_{01}, \eta_{11}, \eta_{21}$ represent the state variables and $\eta_{02}, \eta_{12}, \eta_{22}$ are the co-state variables. At steady state the state and co-state variables are related by

$$\begin{aligned} \eta_{02}(t) &= P_s \eta_{01}(t) \\ \eta_{12}(t) &= P_{f1} \eta_{11}(t) \\ \eta_{22}(t) &= P_{f2} \eta_{21}(t) \end{aligned} \tag{21}$$

where P_s, P_{f1}, P_{f2} satisfy the independent, reduced-order, pure-slow and pure-fast, algebraic Riccati equations. The algebraic Riccati equations are derived from (19)–(21) as

$$\begin{aligned} P_s a_1 - a_4 P_s - a_3 + P_s a_2 P_s &= 0 \\ P_{f1} b_1 - b_4 P_{f1} - b_3 + P_{f1} b_2 P_{f1} &= 0 \\ P_{f2} c_1 - c_4 P_{f2} - c_3 + P_{f2} c_2 P_{f2} &= 0. \end{aligned} \tag{22}$$

The pure-slow and pure-fast algebraic Riccati equations obtained are nonsymmetric. However, their $O(\epsilon)$ perturbations are symmetric ones, that is

$$P_s = P_0 + O(\epsilon), \quad P_{f1} = P_1 + O(\epsilon), \quad P_{f2} = P_2 + O(\epsilon) \tag{23}$$

with

$$\begin{aligned} P_0 A_s + A_s^T P_0 + Q_s - P_0 S_s P_0 &= 0 \\ P_1 A_{11} + A_{11}^T P_1 + Q_{11} - P_1 S_{11} P_1 &= 0 \\ P_2 A_{22} + A_{22}^T P_2 + Q_{22} - P_2 S_{22} P_2 &= 0 \end{aligned} \tag{24}$$

where matrices A_s, Q_s, S_s are defined in (11). The second and third statements in (23) follow directly by examining coefficients $b_j, c_j, j = 1, 2, 3, 4$. Namely, the coefficients of the corresponding algebraic Riccati equations in (22) and (24) are $O(\epsilon)$ apart, that is

$$\begin{aligned} \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} &= D_{11} = T_{11} + O(\epsilon) = \begin{bmatrix} A_{11} & -S_{11} \\ -Q_{11} & -A_{11}^T \end{bmatrix} + O(\epsilon) \\ \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} &= D_{22} = T_{22} + O(\epsilon) = \begin{bmatrix} A_{22} & -S_{22} \\ -Q_{22} & -A_{22}^T \end{bmatrix} + O(\epsilon). \end{aligned}$$

The first statement in (23) is based on the fact that from (20) we have

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = T_{00} - T_{01}L_1^{(0)} - T_{02}L_2^{(0)} + T_{02}L_3^{(0)}L_1^{(0)} + O(\epsilon).$$

Since from (15) $L_1^{(0)} = T_{11}^{-1}T_{10}$, $L_2^{(0)} = T_{22}^{-1}T_{20}$, $L_3^{(0)} = 0$, we get

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = T_{00} - T_{01}T_{11}^{-1}T_{10} - T_{02}T_{22}^{-1}T_{20} + O(\epsilon)$$

which by (11) implies

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = \begin{bmatrix} A_s & -S_s \\ -Q_s & -A_s^T \end{bmatrix} + O(\epsilon).$$

The unique positive semidefinite stabilizing solutions of the algebraic Riccati equations defined in (24) exist under Assumption 2. Then, in view of (23) and by the Implicit Function Theorem the unique solutions of the algebraic Riccati (22) exist. These solutions can be obtained by using the Newton method since (24) produce excellent initial guesses. It is known that the Newton method converges quadratically and that for good initial guesses it requires only four to five iterations. The Newton method for solving the nonsymmetric algebraic Riccati (22) is given by

$$\begin{aligned} &P_s^{(i+1)}\left(a_1 + a_2 P_s^{(i)}\right) - \left(a_4 - P_s^{(i)} a_2\right) P_s^{(i+1)} \\ &\quad = a_3 + P_s^{(i)} a_2 P_s^{(i)}, \qquad P_s^{(0)} = P_0 \\ &P_{f1}^{(i+1)}\left(b_1 + b_2 P_{f1}^{(i)}\right) - \left(b_4 - P_{f1}^{(i)} b_2\right) P_{f1}^{(i+1)} \\ &\quad = b_3 + P_{f1}^{(i)} b_2 P_{f1}^{(i)}, \qquad P_{f1}^{(0)} = P_1 \\ &P_{f2}^{(i+1)}\left(c_1 + c_2 P_{f2}^{(i)}\right) - \left(c_4 - P_{f2}^{(i)} c_2\right) P_{f2}^{(i+1)} \\ &\quad = c_3 + P_{f2}^{(i)} c_2 P_{f2}^{(i)}, \qquad P_{f2}^{(0)} = P_2, \qquad i = 0, 1, 2, \cdots. \end{aligned} \tag{25}$$

In the following we establish the relation between the new and original coordinates and the relation between the solution of the global algebraic Riccati (5) and the solutions of the pure-slow and pure-fast, reduced-order, independent, algebraic Riccati equations (22).

The relationship between the original and new coordinates can be established as follows. Define the permutation matrix as

$$E_2 = \begin{bmatrix} I_{n_0} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n_0} & 0 & 0 \\ 0 & I_{n_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{n_2} & 0 \\ 0 & 0 & I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{n_2} \end{bmatrix}. \tag{26}$$

Then, the new state/co-state variables are related to the old ones by

$$\begin{aligned} \begin{bmatrix} \eta_{01}(t) \\ \eta_{02}(t) \\ \eta_{11}(t) \\ \eta_{12}(t) \\ \eta_{21}(t) \\ \eta_{22}(t) \end{bmatrix} &= E_2^T K E_1 \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} = \Pi \begin{bmatrix} x(t) \\ p(t) \end{bmatrix} \\ &= \begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_3 & \Pi_4 \end{bmatrix} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix}. \end{aligned} \tag{27}$$

In order to establish the relationship between the solutions of the global and local Riccati equations, we first observe that due to the fact that $p(t) = Px(t)$, it follows from (27) that

$$\begin{aligned} \begin{bmatrix} \eta_{01}(t) \\ \eta_{11}(t) \\ \eta_{21}(t) \end{bmatrix} &= (\Pi_1 + \Pi_2 P)x(t) \\ \begin{bmatrix} \eta_{02}(t) \\ \eta_{12}(t) \\ \eta_{22}(t) \end{bmatrix} &= (\Pi_3 + \Pi_4 P)x(t). \end{aligned} \tag{28}$$

Since

$$\begin{bmatrix} \eta_{02}(t) \\ \eta_{12}(t) \\ \eta_{22}(t) \end{bmatrix} = \begin{bmatrix} P_s & 0 & 0 \\ 0 & P_{1f} & 0 \\ 0 & 0 & P_{2f} \end{bmatrix} \begin{bmatrix} \eta_{01}(t) \\ \eta_{11}(t) \\ \eta_{21}(t) \end{bmatrix} \tag{29}$$

the last two formulas imply

$$\begin{bmatrix} P_s & 0 & 0 \\ 0 & P_{1f} & 0 \\ 0 & 0 & P_{2f} \end{bmatrix} = (\Pi_3 + \Pi_4 P)(\Pi_1 + \Pi_2 P)^{-1}. \tag{30}$$

It is shown in [24] that the matrix inversion in (30) exists for small values of singular perturbation parameters. Similarly, we can express P in terms of P_s, P_{f1}, P_{f2}

$$P = \left(\Omega_3 + \Omega_4 \begin{bmatrix} P_s & 0 & 0 \\ 0 & P_{f1} & 0 \\ 0 & 0 & P_{f2} \end{bmatrix} \right) \cdot \left(\Omega_1 + \Omega_2 \begin{bmatrix} P_s & 0 & 0 \\ 0 & P_{f1} & 0 \\ 0 & 0 & P_{f2} \end{bmatrix} \right)^{-1} \tag{31}$$

where

$$\Omega = \begin{bmatrix} \Omega_1 & \Omega_2 \\ \Omega_3 & \Omega_4 \end{bmatrix} = \Pi^{-1}. \tag{32}$$

Invertibility of the matrices in (30) and (31) is established for small values of singular perturbation parameters in [24]. Invertibility of matrix Π can be easily shown.

The efficiency of the proposed technique is demonstrated in [24] on a power system example whose model is presented in [1]. In addition, the results of this paper are extended in [24] to the Kalman filtering multimodeling structure. Those results are successfully applied to a passenger car under unevenness of the road disturbances model of [19] in the context of the singularly perturbed multiparameter Kalman filtering problem [24].

Remark: This paper uses the same technique (the block diagonalization of the Hamiltonian matrix [17] as the paper [25]. However, the problem considered in this paper is more challenging. The method of [25] is based on the Chang transformation in [26] and its application to the state-costate equations of the linear-quadratic optimal control problem. If we intend to apply the Chang transformation to the state-costate equations of this paper we will first have to simplify the problem and assume that it is a single parameter singular perturbation problem, that is $\epsilon_1 = \epsilon_2 = \epsilon$. The Chang transformation will completely decouple the slow subsystem from the fast subsystems, but the fast subsystems will be coupled despite the fact that originally they are coupled only indirectly through the slow subsystem. This is obvious from the application of the nonsingular state transformation (Chang) to (9), which will replace zero matrices in (9) by nonzero elements. This will cause coupling between the fast state-costate variables in (19). Having obtained coupled state-costate variables in (19) produces coupled fast algebraic Riccati equations in (22). That is why in this paper we have used a much more complex transformation of [21] that in addition of extracting independent slow state-costate variables *also completely decouples* fast state-costate variables of the fast subsystems corresponding to the time scales induced by parameters ϵ_1 and ϵ_2. Hence, the proposed method produces further simplifications and introduces full parallelism and decomposition among three subsystems.

REFERENCES

[1] H. Khalil and P. Kokotovic, “Control strategies for decision makers using different models of the same system,” *IEEE Trans. Automat. Contr.*, vol. AC-23, pp. 289–298, 1978.

[2] U. Ozguner, “Near-optimal control of composite systems: The multi time-scale approach,” *IEEE Trans. Automat. Contr.*, vol. AC-24, pp. 652–6559, 1979.

[3] H. Khalil, “Multi-model design of a Nash strategy,” *J. Optim. Theory Appl.*, vol. 31, pp. 553–564, 1980.

[4] P. Kokotovic, “Subsystems, time scales, and multimodeling,” *Automatica*, vol. 17, pp. 789–795, 1981.

[5] V. Saksena and J. Cruz, “A multimodel approach to stochastic Nash games,” *Automatica*, vol. 17, pp. 295–305, 1981.

[6] ——, “Nash strategies in decentralized control of multiparameter singularly perturbed large scale systems,” *Large Scale Syst.*, vol. 2, pp. 219–234, 1981.

[7] V. Saksena and T. Basar, “A multimodel approach to stochastic team problems,” *Automatica*, vol. 18, pp. 713–720, 1982.

[8] V. Saksena, J. Cruz, W. Perkins, and T. Basar, “Information induced multimodel solution in multiple decision maker problems,” *IEEE Trans. Automat. Contr.*, vol. AC-28, pp. 716–728, 1983.

[9] Z. Gajic and H. Khalil, “Multimodel strategies under random disturbances and imperfect partial observations,” *Automatica*, vol. 22, pp. 121–125, 1986.

[10] Z. Gajic, “The existence of a unique and bounded solution of the algebraic Riccati equation of the multimodel estimation and control problems,” *Syst. Contr. Lett.*, vol. 10, pp. 185–190, 1988.

[11] ——, “Quasidecentralized multimodel estimation of linear systems,” in *Analysis and Optimization of Systems*, A. Bensoussan and J. Lions, Eds. New York: Springer Verlag, 1988, pp. 557–568.

[12] A. Vaz and E. Davison, “Modular model reduction for interconnected systems,” *Automatica*, vol. 26, pp. 251–261, 1990.

[13] J. Zhuang and Z. Gajic, “Stochastic multimodel strategy with perfect measurements,” *Control-Theory Adv. Technol.*, vol. 7, pp. 173–182, 1991.

[14] Z. Gajic, “On the quasidecentralized estimation and control of linear stochastic systems,” *Syst. Contr. Lett.*, vol. 8, pp. 441–444, 1987.

[15] Z. Gajic, D. Petkovski, and N. Harkara, “The recursive algorithm for the optimal static output feedback control problem of linear singularly perturbed systems,” *IEEE Trans. Automat. Contr.*, vol. AC-34, pp. 465–468, 1989.

[16] D. Skataric and Z. Gajic, “Linear control of nearly singularly perturbed hydro power plants,” *Automatica*, vol. 28, pp. 159–163, 1992.

[17] Z. Gajic and X. Shen, *Parallel Algorithms for Optimal Control of Large Scale Linear Systems*. New York: Springer Verlag, 1993.

[18] K. Mizukami and F. Suzumura, “Closed-loop Stackelberg strategies for singularly perturbed systems: The recursive approach,” *Int. J. Syst. Sci.*, vol. 24, pp. 887–900, 1993.

[19] M. Salman, A. Lee, and N. Boustany, “Reduced order design of active suspension control,” *ASME Trans. J. Dynamic Syst., Meas., Contr.*, vol. 112, pp. 604–610, 1990.

[20] H. Kwakernaak and R. Sivan, *Linear Optimal Control Systems*. New York: Wiley, 1972.

[21] G. Ladde and S. Rajalakshmi, “Diagonalization and stability of multi-time scale singularly perturbed linear systems,” *Appl. Math. Computation*, vol. 16, pp. 115–140, 1985.

[22] Z. Gajic and M. Qureshi, *Lyapunov Matrix Equation in System Stability and Control*. San Diego, CA: Academic, 1995.

[23] J. Ortega and W. Rheinboldt, *Iterative Solution of Nonlinear Equations on Several Variables*. New York: Academic, 1970.

[24] C. Coumarbatch, Ph.D. dissertation, Rutgers Univ., Dept. Mathematics, to be published.

[25] W. Su, Z. Gajic, and X. Shen, “The exact slow–fast decomposition of the algebraic Riccati equation of singularly perturbed systems,” *IEEE Trans. Automat. Contr.*, vol. 37, pp. 1456–1459, 1992.

[26] K. Chang, “Singular perturbations of a general boundary value problem,” *SIAM J. Math. Anal.*, vol. 3, pp. 520–526, 1972.