

Fig. 3. Popov plot for  $G(s)$ .

criteria, for instance, the Popov criterion and the circle criterion. Our result improves the one presented in [6], where the stability condition was reduced to a large scale linear programming problem. Here, based on the convexity properties of the functions and sets involved, it was possible to propose an efficient algorithm which determines the optimal multiplier, i.e., the one which guarantees the less conservative result. It is based on the solution of a concave nonlinear programming problem which avoids the necessity of including frequency samples and approximates iteratively the optimal multiplier using a time sample allocation policy.

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## The Successive Approximation Procedure for Finite-Time Optimal Control of Bilinear Systems

Zijad Aganovic and Zoran Gajic

**Abstract**—It is shown in this paper that the successive approximation procedure simplifies computations of the optimal solution of a bilinear-quadratic optimal control problem. On the contrary of the results of Hofer and Tibken where the optimal solution has been obtained in terms of a sequence of the differential Riccati equations, in the presented method only solutions of a sequence of the differential Lyapunov equations are required. A chemical reactor example is used to demonstrate the efficiency of the new method.

#### I. INTRODUCTION

From the practical point of view there is a need for the application-oriented controller design technique for bilinear systems. For a bilinear system with a standard quadratic cost functional, with the exception of the simplest cases, however, it is not possible to express the optimal control in the feedback form. Most of the obtained results rely on quadratic cost functionals modified by inclusions of additional nonnegative state-dependent penalizing functions. An overview of the available results can be found in [1]–[15]. The obtained optimal controls have problems with global stabilization of the closed-loop system and with physical meaning of the modified cost functionals.

A new line of thought has been the development of an approximate procedure for the optimal control of bilinear systems [6]. The obtained algorithm is characterized by the explicit linear control law. Since the procedure of the actual computation of the approximate control is still numerically complicated, it is the purpose of this paper to present a new iterative scheme that produces linear control law which is simpler to compute than the one obtained in [6].

#### II. BILINEAR-QUADRATIC OPTIMAL CONTROL

Consider the optimal control problem of a bilinear system

$$\dot{x} = Ax + Bu + \{xN\}u,$$

$$x(t_0) = x_0, \quad \{xN\} = \sum_{j=1}^n x_j N_j \quad (1)$$

where  $x \in R^n$  are the system state variables,  $u \in R^m$  are the control inputs, and  $A$ ,  $B$ , and  $N_j$  are constant matrices of appropriate dimensions with  $N_j \in R^{n \times m}$ . The quadratic cost functional associated with (1) is given by

$$J = \frac{1}{2}x(t_f)^T Fx(t_f) + \frac{1}{2} \int_{t_0}^{t_f} (x^T Qx + u^T Ru) dt \quad (2)$$

where  $Q$  and  $F$  are positive semidefinite symmetric  $n \times n$  matrices and  $R$  is a positive definite symmetric  $m \times m$  matrix.

The application of the minimum principle leads to the following nonlinear two-point boundary value system

$$\begin{aligned} \dot{x}_i &= [Ax]_i - [(B + \{xN\})R^{-1}(B + \{xN\})^T p]_i, & x_i(t_0) &= x_i^0 \\ \dot{p}_i &= -[Qx]_i + [A^T p]_i - \frac{1}{2}p^T \{N_i R^{-1}(B + \{xN\})^T \\ &\quad + (B + \{xN\})R^{-1}N_i^T\} p, & p_i(t_f) &= [Fx(t_f)]_i \end{aligned} \quad (3)$$

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The authors are with Rutgers University, Department of Electrical and Computer Engineering, Piscataway, NJ 08855-0909 USA.

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where  $[\cdot]_i, i = 1, \dots, n$ , stands for the  $i$ th component of the corresponding vector. Unfortunately, there is no analytical solution to this nonlinear two-point boundary value problem. Therefore, there is a need for finding the approximate methods for solving the optimal control problem of bilinear systems.

The work of [6] introduces the iterative scheme that stays in close proximity to the Riccati approach of the linear-quadratic optimization. Namely, the state-costate system (3) is rewritten in the same form as in the linear case

$$\begin{aligned} \dot{x} &= \tilde{A}x - \tilde{B}R^{-1}\tilde{B}^T p, \quad x(t_0) = x^0 \\ \dot{p} &= -\tilde{Q}x - \tilde{A}^T p, \quad p(t_f) = Fx(t_f) \end{aligned} \quad (4)$$

where newly introduced time-varying matrices  $\tilde{A}, \tilde{Q}, \tilde{B}R^{-1}\tilde{B}^T$  are represented by the expressions

$$\begin{aligned} \tilde{A}_{ij} &:= A_{ij} - \frac{1}{2}[(N_j R^{-1} B^T + B R^{-1} N_j^T)p]_i, \quad i, j = 1, \dots, n \\ \tilde{Q}_{ij} &:= Q_{ij} - \frac{1}{2}p^T (N_i R^{-1} N_j^T + N_j R^{-1} N_i^T)p, \quad i, j = 1, \dots, n \\ \tilde{B}R^{-1}\tilde{B}^T &:= (B + \{xN\})R^{-1}(B + \{xN\})^T \\ &\quad - \frac{1}{2}(\{xN\}R^{-1}B^T + B R^{-1}\{xN\}^T). \end{aligned} \quad (5)$$

Using (5) and denoting the iteration index by  $k = 0, 1, \dots$ , and taking into account that

$$\tilde{A}^{(k)} = \tilde{A}(p^{(k)}(t)), \quad \tilde{Q}^{(k)} = \tilde{Q}(p^{(k)}(t))$$

$$\tilde{B}^{(k)}R^{-1}\tilde{B}^{(k)T} = \tilde{B}(x^{(k)}(t))R^{-1}\tilde{B}^T(x^{(k)}(t)) \quad (6)$$

the iterative solution of the state-costate equation (4) can be obtained as [6]

$$\dot{x}^{(k+1)} = \tilde{A}^{(k)}x^{(k+1)} - \tilde{B}^{(k)}R^{-1}\tilde{B}^{(k)T}p^{(k+1)}, \quad x^{(k+1)}(t_0) = x^0 \quad (7a)$$

$$\dot{p}^{(k+1)} = -\tilde{Q}^{(k)}x^{(k+1)} - \tilde{A}^{(k)T}p^{(k+1)}, \quad p^{(k+1)}(t_f) = Fx^{(k+1)}(t_f). \quad (7b)$$

The iteration steps in (7) are carried out by using the Riccati formalism, that is

$$\begin{aligned} \dot{K}^{(k+1)} &= -\tilde{Q}^{(k)} - K^{(k+1)}\tilde{A}^{(k)} - \tilde{A}^{(k)T}K^{(k+1)} \\ &\quad + K^{(k+1)}\tilde{B}^{(k)}R^{-1}\tilde{B}^{(k)T}K^{(k+1)}, \quad K^{(k+1)}(t_f) = F \end{aligned}$$

$$\dot{x}^{(k+1)} = [\tilde{A}^{(k)} - \tilde{B}^{(k)}R^{-1}\tilde{B}^{(k)T}K^{(k+1)}]x^{(k+1)}, \quad x^{(k+1)}(t_0) = x^0. \quad (8)$$

Then, for each iteration step, the linear controller is obtained as [6].

$$u^{(k+1)}(t) = -R^{-1}\tilde{B}^{(k)T}K^{(k+1)}(t)x^{(k+1)}(t) \quad (9)$$

where the gain matrix  $K^{(k+1)}(t)$  has to be calculated iteratively from the Riccati matrix differential equation (8). It was proven in [6] that convergence of this iterative scheme is guaranteed under the following assumption.

*Assumption 1:* The control penalty matrix  $R$  is large enough.

### III. SUCCESSIVE APPROXIMATION APPROACH

The method of successive approximations is the main tool in solving the functional equation of dynamic programming [7]–[9]. It has been used in several control theory papers, for example [10]–[15]. This method can be used as a very powerful decomposition technique which simplifies computations. The monotonicity of successive approximations can be easily established as shown in [7, p. 171]. Proving the convergence, however, is a much more complex task. In the work of Mil'shtein [15], an approximate convergent method for synthesis of the optimal control system is investigated. The approach is based on a combination of the ideas of Lyapunov's second method and Bellman's method of successive approximations. Convergent suboptimal control sequences were also obtained in [10]–[11] and [13]–[14].

The first step in developing a new optimization algorithm is based on the application of the method of successive approximations to the approximative procedure presented in [6]. The idea is to use only one iteration of the successive approximation iterations at each step of the optimization procedure of [6]. As a consequence of this we will have to solve only one differential Lyapunov equation at each iteration step [16]. The convergence proof of the new iterative scheme will be given in the next section. Here, we present the algorithm only.

Equations defined in (7) correspond to the following linear-quadratic finite-time time-varying control problem

$$\dot{x}^{(k+1)} = \tilde{A}^{(k)}x^{(k+1)} + \tilde{B}^{(k)}u^{(k+1)}, \quad x^{(k+1)}(t_0) = x^0 \quad (10)$$

$$\begin{aligned} J^{(k+1)} &= \frac{1}{2}x^{(k+1)T}(t_f)Fx^{(k+1)}(t_f) \\ &\quad + \frac{1}{2}\int_{t_0}^{t_f} (x^{(k+1)T}\tilde{Q}^{(k)}x^{(k+1)} + u^{(k+1)T}Ru^{(k+1)}) dt. \end{aligned} \quad (11)$$

The one-step application of the successive approximation technique to (10)–(11), results in the algorithm [16]

$$\begin{aligned} \dot{x}^{(k+1)} &= [\tilde{A}^{(k)} - \tilde{B}^{(k)}R^{-1}\tilde{B}^{(k)T}P^{(k)}]x^{(k+1)} = A^{(k)}x^{(k+1)}, \\ x^{(k+1)}(t_0) &= x^0 \end{aligned} \quad (12)$$

$$\dot{P}^{(k+1)} + P^{(k+1)}A^{(k)} + A^{(k)T}P^{(k+1)} + Q^{(k)} = 0, \quad P^{(k+1)}(t_f) = F \quad (13)$$

where

$$\begin{aligned} A^{(k)} &= \tilde{A}^{(k)} - \tilde{B}^{(k)}R^{-1}\tilde{B}^{(k)T}P^{(k)} = \tilde{A}^{(k)} - \tilde{S}^{(k)}P^{(k)} \\ Q^{(k)} &= \tilde{Q}^{(k)} + P^{(k)}\tilde{B}^{(k)}R^{-1}\tilde{B}^{(k)T}P^{(k)} = \tilde{Q}^{(k)} + P^{(k)}\tilde{S}^{(k)}P^{(k)}. \end{aligned} \quad (14)$$

For the first iteration step  $k = 0$ , the matrices  $\tilde{A}^{(0)}, \tilde{B}^{(0)}R^{-1}\tilde{B}^{(0)T}$ , and  $\tilde{Q}^{(0)}$  are calculated by using the solution of

$$\dot{x}^{(0)} = (A - BR^{-1}B^T P^{(0)})x^{(0)}, \quad x^{(0)}(t_0) = x^0$$

$$\dot{P}^{(0)} + P^{(0)}A + A^T P^{(0)} + Q = 0, \quad P^{(0)}(t_f) = F \quad (15)$$

which corresponds to the linear part  $\dot{x} = Ax + Bu$  of the bilinear system (1).

Thus, the one-step application of the successive approximations requires the iterative solution of the time-varying differential Lyapunov equations, unlike [6] where the solution of the differential time-varying Riccati equations is required at each iteration.

The approximate control law is stabilizable and given by

$$u^{(k)}(x^{(k+1)}) = -R^{-1}\{\tilde{B}^{(k)}\}^T P^{(k+1)}x^{(k+1)}.$$

It is important to notice that in the proposed scheme we have to solve only one Lyapunov differential equation at each iteration.

Namely, after obtaining the solution of the first Lyapunov differential equation we update all coefficients and go to the next iteration with respect to  $k$ . In that respect, the proposed method is a combination of the successive approximations and the scheme developed by Hofer and Tibken. In the next step, we have to prove the convergence of the proposed method. The proof is along the lines of [6] taking into account the specific features of the successive approximations.

#### IV. PROOF OF CONVERGENCE

In the first part of this proof the expressions for the differences of  $x^{(k+1)}(t) - x^{(k)}(t)$  and  $P^{(k+1)}(t) - P^{(k)}(t)$  will be derived.

From (12) and (14) it can be obtained

$$\frac{d}{dt}(x^{(k+1)} - x^{(k)}) = A^{(k)}(x^{(k+1)} - x^{(k)}) + (A^{(k)} - A^{(k-1)})x^{(k)}. \quad (16)$$

By using the variation of constants method for solving differential equations and the definition the system transition matrix

$$\begin{aligned} \frac{d}{dt}\phi^{(k+1)} &= A^{(k)}\phi^{(k+1)}, \quad \phi^{(k+1)}(0) = I, \\ x^{(k+1)}(t) &= \phi^{(k+1)}(t)x^0 \end{aligned} \quad (17)$$

the expression for the difference  $x^{(k+1)} - x^{(k)}$  can be written as

$$\begin{aligned} x^{(k+1)}(t) - x^{(k)}(t) &= \phi^{(k+1)}(t) \int_{t_0}^t \phi^{(k+1)^{-1}}(s) (A^{(k)}(s) - A^{(k-1)}(s)) \phi^{(k)}(s) x^0 ds. \end{aligned} \quad (18)$$

Similarly, from (13)–(14), it can be shown that

$$\begin{aligned} \frac{d}{dt}(P^{(k+1)} - P^{(k)}) + (P^{(k+1)} - P^{(k)})A^{(k)} + A^{(k)T}(P^{(k+1)} - P^{(k)}) \\ + Q^{(k)} - Q^{(k-1)} + P^{(k)}(A^{(k)} - A^{(k-1)}) \\ + (A^{(k)} - A^{(k-1)})^T P^{(k)} = 0 \end{aligned} \quad (19)$$

so that the corresponding difference is

$$\begin{aligned} P^{(k+1)}(t) - P^{(k)}(t) &= (\phi^{(k+1)^{-T}}(t)) \left\{ \int_{t_0}^t \phi^{(k+1)T}(s) [(Q^{(k)}(s) - Q^{(k-1)}(s)) \right. \\ &\quad + P^{(k)}(s)(A^{(k)}(s) - A^{(k-1)}(s)) + (A^{(k)}(s) - A^{(k-1)}(s)) \\ &\quad \cdot P^{(k+1)}(s)] \phi^{(k+1)}(s) ds \Big\} \phi^{(k+1)^{-1}}(t). \end{aligned} \quad (20)$$

Taking the norm of both sides of (18) and (20), we get

$$\begin{aligned} \|x^{(k+1)}(t) - x^{(k)}(t)\| &\leq \int_{t_0}^t \alpha_1 \|A^{(k)}(s) - A^{(k-1)}(s)\| ds \\ \|P^{(k+1)}(t) - P^{(k)}(t)\| &\leq \int_{t_0}^t \{ \beta_1 \|A^{(k)}(s) - A^{(k-1)}(s)\| \\ &\quad + \beta_2 \|Q^{(k)}(s) - Q^{(k-1)}(s)\| \} ds \end{aligned} \quad (21)$$

where  $\alpha_1, \beta_1, \beta_2$  are obtained by straightforward calculation from (18) and (20). For example  $\alpha_1$  is given by

$$\alpha_1 = \|x^0\| \|\phi^{(k+1)}(t)\phi^{(k+1)^{-1}}(s)\| \|\phi^{(k)}(s)\|. \quad (22)$$

The next step is to estimate the norms of  $\|A^{(k)}(t) - A^{(k-1)}(t)\|$  and  $\|Q^{(k)}(t) - Q^{(k-1)}(t)\|$  in terms of  $\|x^{(k)}(t) - x^{(k-1)}(t)\|$  and  $\|P^{(k)}(t) - P^{(k-1)}(t)\|$ . From (14) the following norm estimates can be obtained

$$\begin{aligned} \|A^{(k)} - A^{(k-1)}\| &\leq \|\tilde{A}^{(k)} - \tilde{A}^{(k-1)}\| + \|\tilde{S}^{(k)} - \tilde{S}^{(k-1)}\| \|P^{(k)}\| \\ &\quad + \|\tilde{S}^{(k-1)}\| \|P^{(k)} - P^{(k-1)}\| \end{aligned} \quad (23)$$

$$\begin{aligned} \|Q^{(k)} - Q^{(k-1)}\| &\leq \|\tilde{Q}^{(k)} - \tilde{Q}^{(k-1)}\| + \|P^{(k)} - P^{(k-1)}\| \|\tilde{S}^{(k)} P^{(k)}\| \\ &\quad + \|P^{(k-1)}\| \|\tilde{S}^{(k)} - \tilde{S}^{(k-1)}\| \|P^{(k)}\| \\ &\quad + \|P^{(k-1)}\| \|\tilde{S}^{(k-1)}\| \|P^{(k)} - P^{(k-1)}\|. \end{aligned} \quad (24)$$

The norms of  $\|\tilde{A}^{(k)} - \tilde{A}^{(k-1)}\|$ ,  $\|\tilde{S}^{(k)} - \tilde{S}^{(k-1)}\|$ , and  $\|\tilde{Q}^{(k)} - \tilde{Q}^{(k-1)}\|$  can be estimated in terms of the original problem matrices (1)–(6) so that the results of [6] can be used, that is

$$\begin{aligned} \|\tilde{A}^{(k)} - \tilde{A}^{(k-1)}\| &\leq \left[ \sum_{j=1}^n \left\| \frac{1}{2} (N_j R^{-1} B^T + B R^{-1} N_j^T) \right\|^2 \right]^{1/2} \\ &\quad \times \{ \|P^{(k)}\| \|x^{(k)} - x^{(k-1)}\| + \|P^{(k)} - P^{(k-1)}\| \|x^{(k-1)}\| \} \end{aligned} \quad (25)$$

$$\begin{aligned} \|\tilde{S}^{(k)} - \tilde{S}^{(k-1)}\| &\leq \left[ \sum_{j=1}^n \left\| \frac{1}{2} (N_j R^{-1} B^T + B R^{-1} N_j^T) \right\|^2 \right]^{1/2} \\ &\quad \times \|x^{(k)} - x^{(k-1)}\| \\ &\quad + \left[ \sum_{i,j=1}^n \|N_i R^{-1} N_j^T\|^2 \right]^{1/2} \\ &\quad \times (\|x^{(k)}\| + \|x^{(k-1)}\|) \|x^{(k)} - x^{(k-1)}\| \end{aligned} \quad (26)$$

$$\begin{aligned} \|\tilde{Q}^{(k)} - \tilde{Q}^{(k-1)}\| &\leq \left[ \sum_{i,j=1}^n \left\| \frac{1}{2} (N_i R^{-1} N_j^T + N_j R^{-1} N_i^T) \right\|^2 \right]^{1/2} \\ &\quad \times (\|P^{(k)} x^{(k)}\| + \|P^{(k-1)} x^{(k-1)}\|) \\ &\quad \times \{ \|P^{(k)}\| \|x^{(k)} - x^{(k-1)}\| \\ &\quad + \|P^{(k)} - P^{(k-1)}\| \|x^{(k-1)}\| \}. \end{aligned} \quad (27)$$

Application of the results of (23)–(27) to (21) leads to the same fixed-point problem as one obtained in [6]

$$\begin{bmatrix} \|x^{(k+1)} - x^{(k)}\| \\ \|P^{(k+1)} - P^{(k)}\| \end{bmatrix} \leq M \begin{bmatrix} \|x^{(k)} - x^{(k-1)}\| \\ \|P^{(k)} - P^{(k-1)}\| \end{bmatrix} \quad (28)$$

where the  $2 \times 2$  matrix  $M$  has to have both eigenvalues inside the unit circle to assure convergence. It can be seen from (25)–(27) that the term  $\|R^{-1}\|$  can be factored out on the right-hand sides of all upper bounds given in (25)–(27) leading to

$$M = M_1(t) \|R^{-1}\|. \quad (29)$$

It is important to notice that the multiplicative influence of  $R^{-1}$  in (29) makes the eigenvalues of the matrix  $M$  arbitrarily small by choosing  $R$  arbitrarily large. Note that the assumption that the penalty matrix  $R$  is large enough is also the main assumption of [6].

The rest of the convergence proof follows by invoking Theorem 4.1 from [6] which states the contraction property for a pair of operators defined in (28) under the assumption that the eigenvalues of the matrix  $M$  are in the unit circle.

#### V. CASE STUDY: CHEMICAL REACTOR

The new method for the optimal control of bilinear systems is applied to the control of a chemical reactor [6]. The bilinear model of the system is given by

$$A = \begin{bmatrix} 13/6 & 5/12 \\ -50/3 & -8/3 \end{bmatrix}, \quad B = \begin{bmatrix} -1/8 \\ 0 \end{bmatrix}$$

$$N_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The normalized state variables  $x_1$  and  $x_2$  represent temperature and concentration of the initial product of the chemical reaction, respectively. The normalized scalar control  $u$  represents the cooling

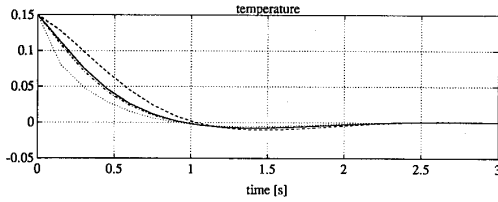


Fig. 1. Profiles of temperature for  $x^0 = (0.15, 0)^T$ .

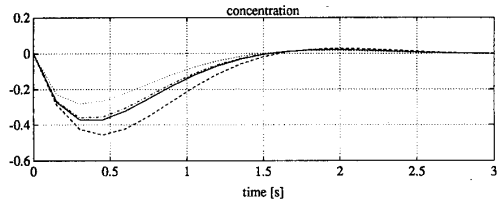


Fig. 2. Profiles of concentration for  $x^0 = (0.15, 0)^T$ .

flow rate in a jacket around the reactor. To transfer the system in finite-time very closely to the steady state given by  $x = 0, u = 0$ , the weighting matrix  $F$  in the performance index has to be chosen dominant compared to the design matrices  $Q$  and  $R$ . A choice of the design matrices  $F, Q$ , and  $R$  is

$$F = \begin{bmatrix} 1000 & 0 \\ 0 & 1000 \end{bmatrix}, \quad Q = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}, \quad R = 1.$$

Initial conditions are  $x(0) = x_0 = [0.15 \ 0]^T$  and the final optimization time  $t_f = 3$ . Simulation results are presented in Figs. 1 and 2 where the solid lines are the optimal trajectories. The approximations of the actual optimal trajectories are represented by the dashed lines for the first approximations, the dotted lines for the second approximations, and the dashed-dotted lines for the third approximations. It can be seen from Figs. 1 and 2 that the new method preserves very good convergence in this particular example. In addition, the convergence is achieved with relatively small value for the control penalty matrix  $R$  so that the constraint imposed in Assumption 1 does not seem to be very severe.

VI. CONCLUSION

In this paper, the new method for the optimization of the bilinear-quadratic control systems is developed. The starting point is the algorithms of [6] for the approximation of the optimal solution of the bilinear optimal control problem. That method itself presents an interesting approach from the application point of view. Namely, the optimization problem of the bilinear (nonlinear) system is replaced by a sequence of the linear optimization problems.

The new algorithm presented simplifies the procedure of [6] by replacing the computation of the solution of the time-varying differential Riccati equation by the problem of solving the differential time-varying Lyapunov equation at each iteration level. The numerical example shows that the speed of convergence of the new algorithm is not inferior to the one of the algorithm from [6].

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