6.2, 6.9, 6.19,
due May 11,

Apr. 15, 94 (11)

Chapter VI

NUMERICAL DETERMINATION OF OPTIMAL TRAJECTORIES

Optimal control leads in general to NONLINEAR TPBVP that can not be solved analytically

t_f fixed

$x(t_f)$ free

$$(1)\quad \dot{x} = \frac{\partial H}{\partial p} = a(x, u, t), \qquad x(t_0) = x_0$$

$$(2)\quad \dot{p} = -\frac{\partial H}{\partial x} = -\left(\frac{\partial a}{\partial x}\right)^T p - \frac{\partial g(x,u,t)}{\partial x}, \qquad p(t_f) = \frac{\partial h}{\partial x}(x(t_f))$$

$$(3)\quad 0 = \frac{\partial H}{\partial u} = \left(\frac{\partial a}{\partial u}\right)^T p + \frac{\partial g}{\partial u} \qquad \Rightarrow u = f(x, p, t)$$

(6.2) THE METHOD OF STEEPEST DESCENT

The steepest Descent Algorithm

1) select a discrete approximation to the nominal control

$$u^{(0)}(t), \quad t \in [t_0, t_f]$$

2) Solve (1)

$$\dot{x}^{(i)} = a(x^{(i)}, u^{(i)}, t), \quad x(t_0) = x_0, \quad i = 0, 1, 2, \ldots$$

3) Solve (2)

$$\dot{p}^{(i)} = -\left[\frac{\partial a(x^{(i)}, u^{(i)}, t)}{\partial x}\right]^T p^{(i)} - \frac{\partial g}{\partial x}(x^{(i)}, u^{(i)}, t)$$

with $p(t_f)$ given

4) Evaluate

$$\frac{\partial H^{(i)}}{\partial u} = \frac{\partial}{\partial u} H(x^{(i)}, p^{(i)}, u^{(i)}), \quad i = 0, 1, 2, \ldots$$

5) If $\left\| \frac{\partial H^{(i)}}{\partial u} \right\| \le \gamma$ small positive number STOP

otherwise

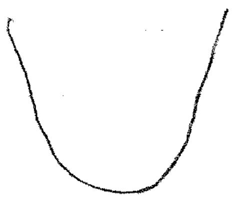
$$\boxed{u^{(i+1)} = u^{(i)} - \tau \frac{\partial H^{(i)}}{\partial u}} \qquad i = 0, 1, 2, \ldots$$

step size

This is a gradient type technique

<u>we are going to the minimum in the negative gradient direction.</u>

THEORY:



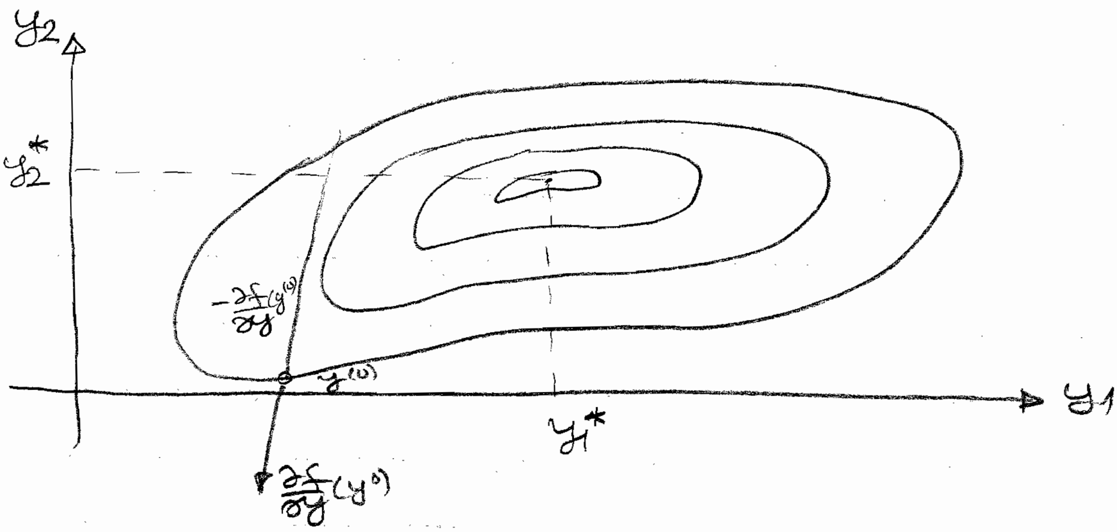
$f(y_1, y_2)$

$$df(y_1, y_2) = \frac{\partial f}{\partial y_1} \Delta y_1 + \frac{\partial f}{\partial y_2} \Delta y_2 = 0$$

$$= \begin{bmatrix} \frac{\partial f}{\partial y_1} \\ \frac{\partial f}{\partial y_2} \end{bmatrix}^T \begin{bmatrix} \Delta y_1 \\ \Delta y_2 \end{bmatrix} = \left(\frac{\partial f}{\partial y}\right)^T \Delta y = 0$$

$$\Rightarrow \boxed{\frac{\partial f}{\partial y} = 0}$$

at the extreme point it is necessary that the gradient is zero



$$y^{(i+1)} = y^{(i)} - \tau \cdot z(y^{(i)}) = y^{(i)} + \Delta y^{(i)}$$

$$z^{(i)} = \frac{\frac{\partial f}{\partial y}^{(i)}}{\left\| \frac{\partial f}{\partial y}^{(i)} \right\|}$$ unit vector in the gradient direction

$\Rightarrow$

$$df = \left(\frac{\partial f}{\partial y}\right)^T \Delta y = -\tau \left\| \frac{\partial f}{\partial y} \right\|^2 \leq 0$$

<u>Applied to our problem:</u>

$$\dot{x}^{(i)} = a(x^{(i)}, u^{(i)}, t) \qquad x^{(i)}(t_0) = x_0$$

$$\dot{p}^{(i)} = -\frac{\partial H}{\partial x}(x^{(i)}, u^{(i)}, p^{(i)}, t) \qquad p^{(i)}(t_f) = \frac{\partial h}{\partial x}(x^{(i)}(t_f))$$

$$\frac{\partial H}{\partial u}(x^{(i)}, u^{(i)}, p^{(i)}, t) = 0$$ for <u>nominal</u> control

If we use notation $\delta x = x^{(i+1)} - x^{(i)}$
$\delta u = u^{(i+1)} - u^{(i)}$, $\delta p = p^{(i+1)} - p^{(i)}$

$$\Rightarrow \delta J_a = \int_{t_0}^{t_f} \left[\frac{\partial H}{\partial u}(x^{(i)}, u^{(i)}, p^{(i)}, t)\right]^T \delta u \cdot dt \qquad (5.3\text{-}8)$$

We want to make δJ_a negative

$$\delta u = u^{(i+1)} - u^{(i)} = -\tau \frac{\partial H}{\partial u}^{(i)}$$

$\Rightarrow$

$$\delta J_a = -\tau \int_{t_0}^{t_f} \left\| \frac{\partial H}{\partial u}^{(i)} \right\|^2 dt \leq 0$$

main issue: δu must be sufficiently small

convergence: the initial guess is not usually crucial.

MAIN FEATURE: Gradient methods are very slow near to minimum because the gradient is very small near a minimum.

(6.3) VARIATION OF EXREMALS

Motivation:

A First Order Optimal Control Problem

$$\dot{x} = a(x(t), u(t), t) \qquad x(t_0) = x_0, \qquad t_0, t_f \text{ fixed}$$

$$J = \int_{t_0}^{t_f} g(x(t), u(t), t)\, dt$$

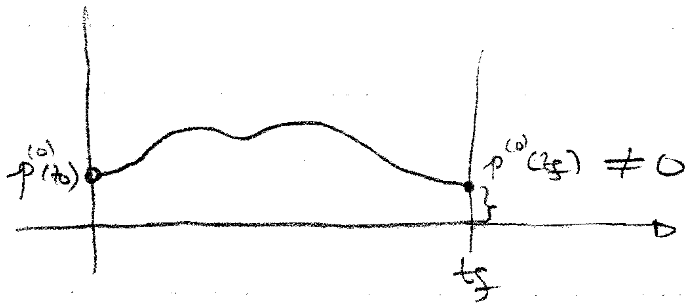
$$\frac{\partial H}{\partial u} = 0 \Rightarrow u(t) = f(x, p, t)$$

$$\dot{x} = a_1(x(t), p(t), t) \qquad x(t_0) = x_0$$

$$\dot{p} = d(x(t), p(t), t) \qquad p(t_f) = 0$$

guess $p^{(0)}(t_0)$

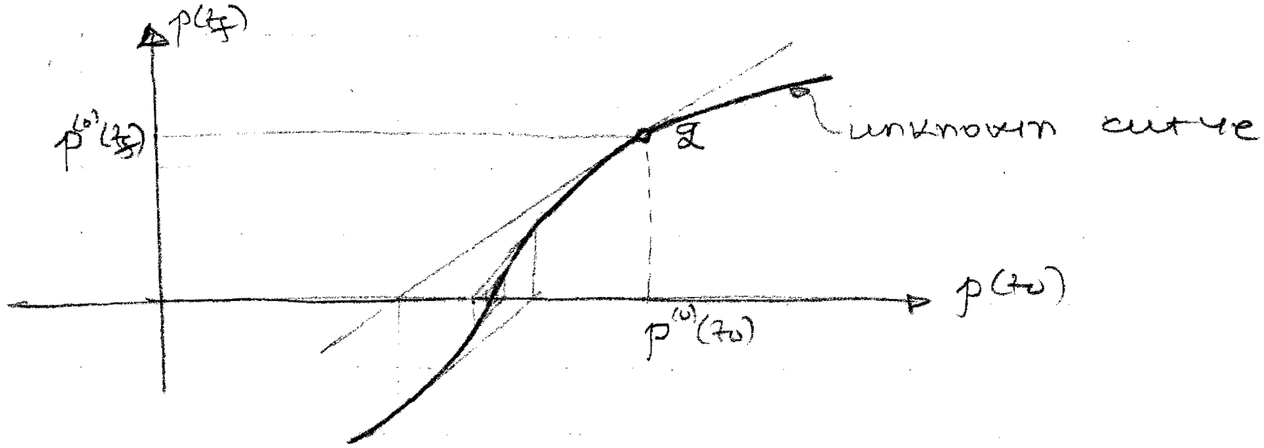
$$\dot{x}^{(0)} = a_1(x^{(0)}(t), p^{(0)}(t), t) \qquad x(t_0) = x_0$$
$$\dot{p}^{(0)} = d(x^{(0)}(t), p^{(0)}(t), t) \qquad \underbrace{p(t_0) = p^{(0)}(t_0)}_{\text{guess}} \Bigg\} \Rightarrow \begin{matrix} x^{(0)}(t) \\ p^{(0)}(t) \end{matrix}$$



$p^{(0)}(t_f)$ is a function of $p^{(0)}(t_0)$

but the analytical expression is not known.

variation of extremals method adjust $p^{(i+1)}(t_0)$ according to $p^{(i)}(t_f)$.



slope at q is $\left.\dfrac{dp(t_f)}{dp(t_0)}\right|_{p^{(0)}(t_0)} \simeq \dfrac{\delta p^{(0)}(t_f)}{\delta p^{(0)}(t_0)} = m$

tangent

$$p(t_f) = m\,p(t_0) + b \implies b = p^{(0)}(t_f) - m\,p^{(0)}(t_0)$$

$$p(t_f) = m\,p(t_0) + [p^{(0)}(t_f) - m\,p^{(0)}(t_0)]$$

<u>need to have $p(t_f) = 0$</u>

$$0 = m\,p^{(1)}(t_0) + [p^{(0)}(t_f) - m p^{(0)}(t_0)]$$

$$p^{(1)}(t_0) = p^{(0)}(t_0) - (m)^{-1}\, p^{(0)}(t_f)$$

or

$$p^{(1)}(t_0) = p^{(0)}(t_0) - \left[\frac{dp(t_f)}{dp(t_0)}\Big|_{p^{(0)}(t_0)}\right]^{-1} p^{(0)}(t_f)$$

Newton method

in general

$$\boxed{p^{(i+1)}(t_0) = p^{(i)}(t_0) - \left[\frac{dp(t_f)}{dp(t_0)}\Big|_{p^{(i)}(t_0)}\right]^{-1} p^{(i)}(t_f)}$$

(Ex)

$$\dot{x} = -2x - p + 6 \qquad x(0) = 3$$

$$\dot{p} = 4x + 3p \qquad p(1) = 0$$

guess $p^{(0)} = 0 \Rightarrow$

$$x^{(0)}(t) = -4e^{-t} - 2e^{2t} + 9$$

$$p^{(0)}(t) = 4e^{-t} + 8e^{2t} - 12$$

perturb $p^{(0)}(0)$ by a small amount $\delta p^{(0)}(0) = 10^{-3}$

$\Rightarrow$

$$p^{(0)}(t) + \delta p^{(0)}(t) = 4e^{-t} + 8e^{2t} - 12 \underbrace{- \frac{10^{-3}}{3} e^{-t} + \frac{0.004}{3} e^{2t}}$$

$$\frac{dp(t_f)}{dp(t_0)}\Big|_{p(t_0)=0} = \frac{-\frac{10^{-3}}{3}e^{-1} + \frac{0.004}{3}e^{2}}{0.001} = -\frac{1}{3}e^{-1} + \frac{4}{3}e^{2}$$

$$\Rightarrow p^{(1)}(t_0) = 0 - \left[-\tfrac{1}{3}e^{-1} + \tfrac{4}{3}e^{2}\right]^{-1} \cdot \left[4e^{-1} + 8e^{2} - 12\right]$$

$$p^{(1)}(t_0) = -4{,}993$$

$$\rightarrow x^{(1)}(t) = -0{,}336e^{2t} - 5{,}664e^{-t} + 9$$
$$p^{(1)}(t) = 1{,}342e^{2t} + 5{,}664e^{-t} - 12$$

EXTENSION FOR SYSTEMS OF $2n$ DIFF. EQS.

$p \in \mathbb{R}^n$
$x \in \mathbb{R}^n$

$$p^{(i+1)}(t_0) = p^{(i)}(t_0) - \left[P_p(p^{(i)}(t_0), t_f)\right]^{-1} p^{(i)}(t_f)$$

$P_p(p^{(i)}(t_0), t_f)$ <u>costate influence function matrix</u>

$$P_p(p^{(i)}(t_0), t) \triangleq \begin{bmatrix} \frac{\partial p_1(t)}{\partial p_1(t_0)} & \frac{\partial p_1(t)}{\partial p_2(t_0)} & \cdots & \frac{\partial p_1(t)}{\partial p_n(t_0)} \\ \vdots & & & \vdots \\ \frac{\partial p_n(t)}{\partial p_1(t_0)} & & \cdots & \frac{\partial p_n(t)}{\partial p_n(t_0)} \end{bmatrix}_{p^{(i)}(t_0)}$$

If $h(x(t_f))$ is present

$$p^{(i+1)}(t_0) = p^{(i)}(t_0) + \left\{\left[\frac{\partial^2 h}{\partial x^2}(x(t_f))\right] P_x(p(t_0), t_f) - P_p(p(t_0), t_f)\right\}^{-1} \cdot$$
$$\cdot \left[p(t_f) - \frac{\partial h}{\partial x}(x(t_f))\right]$$

where $P_x(p(t_0), t_f)$ is the state influence matrix

(text Prob. 6.1)

$$P_x(p^{(i)}(t_0),t) \triangleq \left[\begin{matrix} \frac{\partial x_1(t)}{\partial p_1(t_0)} & \frac{\partial x_1(t)}{\partial p_2(t_0)} & \cdots & \frac{\partial x_1(t)}{\partial p_n(t_0)} \\ \vdots & & & \vdots \\ \frac{\partial x_n(t)}{\partial p_1(t_0)} & \cdots & \cdots & \frac{\partial x_n(t)}{\partial p_n(t_0)} \end{matrix}\right]\Bigg|_{p^{(i)}(t_0)}$$

Also

$$\frac{\partial^2 h}{\partial x^2} = \left\{\frac{\partial^2 h}{\partial x_j \partial x_k}(x(t_f))\right\}_{jk}$$

Determination of the Infuence Function Matrices

(1) Using $p(t_0) = p^{(i)}(t_0)$ and $x(t_0) = x_0$ integrate the reduced state and costate equations from t_0 to t_f

(2) Perturb the first component of the vector $p^{(i)}(t_0)$ by an amount $\delta p_1(t_0)$ and again integrate the reduced state and costate eqs. from t_0 to t_f. The first column of P_p is found from

$$\frac{\partial p(t_f)}{\partial p_1(t_0)}\Bigg|_{p^{(i)}(t_0)} \doteq \frac{\delta p(t_f)}{\delta p_1(t_0)}$$

similarly

$$\frac{\partial x}{\partial p_1(t_0)} \Rightarrow \text{first column in } P_x$$

The equations for the influence matrices

(1) $\dot{x} = \frac{\partial H}{\partial p}$

(2) $\dot{p} = -\frac{\partial H}{\partial x}$

$$\frac{\partial}{\partial p(t_0)}(\dot{x}(t)) = \frac{\partial}{\partial p(t_0)}\left[\frac{\partial H}{\partial p}(x, p, t)\right]$$

$$\frac{\partial}{\partial p(t_0)}(\dot{p}(t)) = \frac{\partial}{\partial p(t_0)}\left[-\frac{\partial H}{\partial x}(x, p, t)\right]$$

$$\frac{d}{dt}\left[\frac{\partial x}{\partial p(t_0)}\right] = \left[\frac{\partial^2 H}{\partial p\,\partial x}(x, p, t)\right]\frac{\partial x}{\partial p(t_0)} + \left[\frac{\partial^2 H}{\partial p^2}(x, p, t)\right]\frac{\partial p(t)}{\partial p(t_0)}$$

$$\boxed{\begin{aligned}\frac{d}{dt}\left[P_x(p^{(i)}(t_0), t)\right] &= \left(\frac{\partial^2 H}{\partial p\,\partial x}\right)_i P_x(p^{(i)}(t_0), t) + \\ &+ \left(\frac{\partial^2 H}{\partial p^2}\right)_i P_p(p^{(i)}(t_0), t)\end{aligned}}$$

$$\boxed{P_x(p^{(i)}(t_0), t_0) = 0} = \frac{\partial x(t_0)}{\partial p(t_0)} = 0$$

Also

$$\boxed{\begin{aligned}\frac{d}{dt}\left[P_p(p^{(i)}(t_0), t)\right] &= -\left(\frac{\partial H^2}{\partial x^2}\right)_i P_x(p^{(i)}(t_0), t) \\ &+ \left(-\frac{\partial^2 H}{\partial x\,\partial p}\right)_i P_p(p^{(i)}(t_0), t)\end{aligned}}$$

$$\boxed{P_p(p^{(i)}(t_0), t_0) = I} = \frac{\partial p(t_0)}{\partial p(t_0)}$$

1) Divergence may result from a poor initial guess

2) Once convergence begins, it is rapid.

(6.4) QUASILINEARIZATION

A sequence of linear TPBVP is solved

SOLUTION OF LINEAR TPBVP:

$$(1)\ \begin{cases} \dot{x} = a_{11}(t)\,x + a_{12}(t)\,p(t) + e_1(t) & x(t_0) = x_0 \\ \dot{p} = a_{21}(t)\,x + a_{22}(t)\,p(t) + e_2(t) & p(t_f) = p_f \end{cases}$$

First solve (numerically) the homogeneous differential equations

$$(2)\quad \begin{array}{ll} \dot{x}^H(t) = a_{11}(t)\,x^H(t) + a_{12}(t)\,p^H(t) & x^H(t_0) = 0 \\ \dot{p}^H(t) = a_{21}(t)\,x^H(t) + a_{22}(t)\,p^H(t) & p^H(t_0) = 1 \end{array} \Bigg\} \text{ convenient choice}$$

with arbitrary initial conditions

Next, find a particular solution of (1) with $x^P(t_0) = x_0$, and $p^P(t_0) = 0$

$$\Longrightarrow \quad \begin{aligned} x(t) &= c_1\,x^H(t) + x^P(t) \\ p(t) &= c_1\,p^H(t) + p^P(t) \end{aligned}$$

Since $p(t_f) = p_f = c_1\,p^H(t_f) + p^P(t_f)$

$$\Longrightarrow \quad c_1 = \frac{p_f - p^P(t_f)}{p^H(t_f)}$$

LINEARIZATION OF THE Reduced State-Costate Equations

$$\frac{\partial H}{\partial u} = 0 \implies u(t) = f(x(t), p(t), t) \implies$$

$$\dot{x} = a(x, p, t) \qquad x(t_0) = x_0$$
$$\dot{p} = d(x, p, t) \qquad p(t_f) = p_f$$

By Taylor series

$$\dot{x} = a(x^{(0)}, p^{(0)}, t) + \frac{\partial a}{\partial x}(x - x^{(0)}) + \frac{\partial a}{\partial p}(p - p^{(0)})$$

$$\dot{p} = p(x^{(0)}, p^{(0)}, t) + \frac{\partial d}{\partial x}(x - x^{(0)}) + \frac{\partial d}{\partial p}(p - p^{(0)})$$

$$\dot{x} = \overbrace{\frac{\partial a}{\partial x}}^{a_{11}(t)} x + \overbrace{\frac{\partial a}{\partial p}}^{a_{12}(t)} p + \overbrace{\left(a(x^{(0)}, p^{(0)}, t) - \frac{\partial a}{\partial x} x^{(0)} - \frac{\partial a}{\partial p} p^{(0)}\right)}^{e_1(t)}$$

$$\dot{p} = \frac{\partial p}{\partial p} p + \frac{\partial p}{\partial x} x + \underbrace{(\qquad\qquad)}_{e_2(t)}$$

Thus,

<u>Step 1</u>. Linearize nonlinear diff. eq

<u>Step 2</u>. Solve linear TBPVP

Opt problem by quasilinearization

$$\dot{x} = a(x, p, t) \qquad x(t_0) = x_0$$
$$\dot{p} = -\frac{\partial H}{\partial x}(x, p, t) \qquad p(t_f) = p_f$$

Linearization $\Rightarrow$

$$\dot{x}^{(i+1)} = a(x^{(i)}, p^{(i)}, t) + \frac{\partial a}{\partial x}(x^{(i)}, p^{(i)}, t)(x^{(i+1)} - x^{(i)}) + \frac{\partial a}{\partial p}(x^{(i)}, p^{(i)}, t)(p^{(i+1)} - p^{(i)})$$

$$\dot{p}^{(i+1)} = -\frac{\partial H}{\partial x}(x^{(i)}, p^{(i)}, t) - \frac{\partial^2 H}{\partial x^2}(x^{(i)}, p^{(i)}, t)(x^{(i+1)} - x^{(i)}) - \frac{\partial^2 H}{\partial x \partial p}(x^{(i)}, p^{(i)}, t)(p^{(i+1)} - p^{(i)})$$

$$\Rightarrow \left.\begin{aligned} \dot{x}^{(i+1)} &= A_{11}(t)\, x^{(i+1)} + A_{12}(t)\, p^{(i+1)} + e_1(t) \\ \dot{p}^{(i+1)} &= A_{21}(t)\, x^{(i+1)} + A_{22}(t)\, p^{(i+1)} + e_2(t) \end{aligned}\right\} \circ$$

generate n-solutions of Homoge. sys

$$\left\{\begin{aligned} \dot{x}^{H(i+1)} &= A_{11}(t)\, x^{H(i+1)} + A_{12}(t)\, p^{H(i+1)}, & x^H(t_0) &= 0 \\ \dot{p}^{H(i+1)} &= A_{21}(t)\, x^{H(i+1)} + A_{22}(t)\, p^{H(i+1)}, & & \end{aligned}\right.$$

with $p^{H_1}(t_0) = [1 \; 0 \; - \; - \; 0]^T$

$p^{H_2}(t_0) = [0 \; 1 \; 0 \; - \; - \; 0]^T$

$\vdots$

$p^{H_n}(t_0) = [0 \; - \; - \; - \; 1]^T$

$$\Rightarrow x^{H_1}(t), \ldots, x^{H_n}(t)$$

next find a particular solution to

$$\dot{x}^{p(i+1)} = A_{11}(t)x^{p(i+1)} + A_{12}p^{p(i+1)} + e_1(t), \quad p^p(t_0) = 0$$

$$\dot{p}^{H(i+1)} = A_{21}(t)x^{p(i+1)} + A_{22}p^{p(i+1)} + e_2(t), \quad x^p(t_0) = x_0$$

The complete solution is in the form

$$x^{(i+1)} = c_1 x^{H_1} + c_2 x^{H_2} + \cdots + c_n x^{H_n} + x^p(t)$$

$$p^{(i+1)} = c_1 p^{H_1} + c_2 p^{H_2} + \cdots + c_n p^{H_n} + p^p(t)$$

$$p_f - p^p(t_f) = [p^{H_1} \mid p^{H_2} \mid \cdots \; p^{H_n}]\, c$$

$$\Rightarrow c = [p^{H_1} \mid p^{H_2} \mid \cdots \mid p^{H_n}] \left(p_f - p^p(t_f) \right)$$

STOP: if $\left\| \begin{matrix} x^{(i+1)} - x^{(i)} \\ p^{(i+1)} - p^{(i)} \end{matrix} \right\| \leq \delta$ small positive number

FEATURES:

1) $x^{(0)}, p^{(0)}$ poor initial guess $\Rightarrow$ divergence

2) converge quadratically in the vicinity of the optimum