HW #5 7, 11, 12

HW #6 2, 3, 13

Apr. 1, 1984, (9)

HW #7: 4, 21, 27, 35

5.5 MINIMUM CONTROL EFFORT PROBLEM

Problem Formulation:

Find a control $u^*(t)$ satisfying

$$-M_{i-} \leq u_i(t) \leq M_{i+}$$

which transfers a system described by

$$\dot{x}(t) = a(x(t), u(t), t)$$

from an arbitrary initial state x_0 to a specified target set $S(t)$ with a minimum expenditure of control effort.

As a measure of control effort we consider

a) $$J_1(u) = \int_{t_0}^{t_f} \left[\sum_{i=1}^{m} \beta_i |u_i(t)| \right] dt$$ minimum fuel problem

b) $$J_2(u) = \int_{t_0}^{t_f} \left[\sum_{i=1}^{m} r_i \, u_i^2(t) \right] dt$$ minimum energy problem

(a) MINIMUM FUEL PROBLEM

For a class of minimum-fuel problems

$$\begin{cases} \dot{x} = a(x(t), t) + B(x(t), t)\, u(t) \\ J = \int_{t_0}^{t_f} \left[\sum_{i=1}^{m} |u_i(t)| \right] dt \end{cases}$$

system linear in $u(t)$

$-1 \leq u_i(t) \leq 1$

$$H(x, u, p, t) = \sum_{i=1}^{m} |u_i(t)| + p^T(t)\left[a(x(t), t) + B(x(t), t)\, u(t)\right]$$

The minimum principle $\Rightarrow$

$$\sum_{i=1}^{m} |u_i^*(t)| + \cancel{p^{*T} a(x^*, t)} + p^{*T} B(x^*, t) u^* \leq$$

$$\sum_{i=1}^{m} |u_i(t)| + \cancel{p^{*T} a(x^*, t)} + p^{*T} B(x^*, t)\, u$$

or

$$\sum_{i=1}^{m} |u_i^*(t)| + p^{*T}(t) B(x^*, t)\, u^* \leq \sum_{i=1}^{m} |u_i(t)| + p^{*T}(t) B(x^*(t), t)\, u$$

Let us express B on the form

$$B(x^*, t) = [\, b_1(x^*, t) \,|\, b_2(x^*, t) \,|\, \ldots \,|\, b_m(x^*, t) \,]$$

$\Rightarrow$

$$|u_i^*(t)| + p^{*T}(t)\, b_i(x^*)\, u_i^* \leq |u_i(t)| + p^{*T}(t)\, b_i(x^*, t)\, u_i$$

$$|u_i(t)| = \begin{cases} u_i(t) & u_i > 0 \\ 0 & u_i(t) = 0 \\ -u_i(t) & u_i(t) < 0 \end{cases}$$

$\Rightarrow$

$$[1 + p^{*T}(t)\, b_i(x^*, t)]\, u_i^* \leq [1 + p^{*T} b_i(x^*, t)]\, u_i \quad , \quad u_i > 0$$

and

$$[-1 + p^{*T}(t)\, b_i(x^*, t)]\, u_i^* \leq [-1 + p^{*T} b_i(x^*, t)]\, u_i \quad , \quad u_i < 0$$

$$u_i^*(t) = \begin{cases} 1 & -\infty < p^{*T}(t)\, b_i(x^*, t) < -1 \\ 0 & -1 < \quad '' \quad < +1 \\ -1 & +1 < \quad '' \quad < +\infty \\ \text{undetermined solution} & p^{*T} b_i(x^*, t) = \pm 1 \end{cases}$$

$\boxed{M_{i+} = 1 \quad M_{i-} = -1}$

singular solution

BANG – OFF – BANG CONTROL

5.6. SINGULAR INTERVALS IN OPTIMAL CONTROL

$$H(x^*, u^*, p^*) \leq H(x^*, u, p^*)$$

If this provides no information about relationship among u^*, p^* and x^* $\Rightarrow$ singular problem

(Ex: 5.6-1)

$$\dot{x}_1 = x_2 \qquad \text{min time problem}$$
$$\dot{x}_2 = u \qquad |u(t)| \leq 1$$

$$H = 1 + p_1 x_2 + p_2 u$$

$$1 + p_1^* x_2^* + p_2^* u^* \leq 1 + p_1^* x_2^* + p_2^* u$$

If $p_2^*(t) = 0$, $\underline{\underline{t \in (t_1, t_2)}}$ $\Rightarrow$ singular problem

$$\dot{p}_1^* = -\frac{\partial H}{\partial x_1} = 0 \Rightarrow p_1^*(t) = c_1$$
$$\dot{p}_2^* = -\frac{\partial H}{\partial x_2} = 0 \Rightarrow p_2^*(t) = -c_1 t + c_2$$

$$p_2^*(t) = 0 \Rightarrow p_1^*(t) = 0 \Rightarrow H^* = 1$$

$\Rightarrow$ no singular intervals

$H^* = 0$ for the final time free problem

also H does not explicitely depends on t.

Singular Intervals in Linear Time-Optimal Problems

$$\dot{x} = Ax + bu \qquad |u(t)| \leq 1.0$$

$$H(x,u,p) = 1 + p^T Ax + p^T bu$$

(1) $1 + p^{*T} Ax^* + p^{*T} bu^* \leq 1 + p^{*T} Ax^* + p^{*T} bu$

the final time is free $\Rightarrow$ $H^* = 0$

(2) $1 + p^{*T} Ax^* + p^{*T} bu^* = 0$

(1) $\Rightarrow$ a singular interval is for $p^{*T}(t) b = 0, \ t \in [t_1, t_2]$

$p^*(t) \neq 0$ since $H^*(p^* = 0) = 1 \neq 0$

thus

$p^*(t) b = 0$ for $t \in [t_1, t_2]$

also

$\frac{d^k}{dt^k} [p^*(t) b] = 0 \qquad k = 1, 2, \ldots$

$\frac{d^k}{dt^k} (p^*(t) b) = \frac{d^k p^*(t)}{dt^k} \cdot b$

$\dot{p}^{*T} = - \frac{\partial H}{\partial x} = -p^{*T} A \Rightarrow \dot{p}^* = -A^T p^* \Rightarrow p^*(t) = e^{-A^T \cdot t} c$

$\frac{d}{dt} p^* b = \underbrace{(-A^T) e^{-A^T t} c}_{} \cdot b = 0 \Rightarrow (e^{-A^T t} c)^T Ab = 0$

$\frac{d^2}{dt^2} p^* = \ldots \Rightarrow (\quad)^T A^2 b = 0$

$p^{(k)*T} b = 0 \Rightarrow (e^{-A^T t} c)^T A^k b = 0$

for $k = n-1$

$$\underbrace{(e^{-A^T t} c)^T}_{p(t) \neq 0} [b \mid Ab \mid A^2 b \mid \dots \mid A^{n-1} b] = 0$$

$\Rightarrow [b \mid Ab \mid A^2 b \mid \dots \mid A^{n-1} b]$ <u>must be singular</u>

For a singular interval to exist, it is necessary that the system is UNCONTROLLABLE

<u>SINGULAR Intervals IN LINEAR Fuel-Optimal Problems</u>

the final time is free $\Rightarrow H = 0$

Ex. 5.6-2

$\dot{x}_1 = x_2$

$\dot{x}_2 = u$

$H(x, u, p) = |u(t)| + p_1 x_2 + p_2 u$

(3) $|u^*| + \cancel{p_1^* x_2^*} + p_2^* u^* \le |u| + \cancel{p_1^* x_2^*} + p_2^* u$

and

(4) $|u^*(t)| + p_1^*(t) x_2^*(t) + p_2^*(t) u^*(t) = 0$

singular interval for

$p_2^* = \pm 1 \overset{(4)}{\Rightarrow} p_1^* x_2^*(t) = 0$

$p_1^*(t) = c_1$

$p_2^*(t) = -c_1 t + c_2 \qquad p_2^*(t) = \pm 1 \Rightarrow c_1 = 0 \Rightarrow p_1^* = 0$

$\Rightarrow p_1^* x_2^* = 0$

$\Rightarrow$ singular interval can exist even if the system is completely controllable

Theory:

$$\dot{x} = Ax + bu \qquad |u(t)| \le 1$$

$$J(u) = \int_0^{t_f} |u(t)|\,dt$$

$$H = |u(t)| + p^T A x + p^T b u$$

$$|u^*| + \cancel{p^* A x^*} + p^{*T} b u^* \le |u| + \cancel{p^* A x^*} + p^* b u$$

singular interval exists if

$$p^{*T}(t)\, b = \pm 1 \qquad \text{also} \quad \frac{d^k}{dt^k}(p^{*T} b) = 0 \quad k = 1, 2, \ldots$$

Also

$$p^{*T} A x^*(t) = 0$$

$$\Rightarrow \quad (e^{-A^T t} c)^T (Ab \mid A^2 b \mid \cdots \mid A^n b) = 0^T$$

or

$$(b \mid Ab \mid \cdots \mid A^{n-1} b)^T A^T \underbrace{e^{-A^T t} c}_{p(t) \neq 0} = 0$$

$$\Rightarrow \quad (b \mid Ab \mid \cdots \mid A^{n-1} b)^T A^T \text{ is singular}$$

Thus a <u>necessary condition</u> for a singular interval to exist is that either the system matrix is singular or the system is not completely controllable.

Use the fact that the Hamiltonian must be zero (final time free and time invariant system – H does not depend explicitly on time).

Since $H^* = 0 \Rightarrow \dot{H}^* = 0$

$$\ddot{H}^* = 0$$

$$\vdots$$

$$H^{(i)*} = 0$$

5.23. $\dot{x}_1 = x_2$

$\dot{x}_2 = x_1 + x_2 u$

$J = \int_0^{2s} |u(t)|\, dt$

Bilinear system

$|u(t)| \le 1$

a) $H = |u(t)| + p_1 x_2 + p_2 (x_1 + x_2 u)$

(1) $-\frac{\partial H}{\partial x_1} = \dot{p}_1 = -p_2$

(2) $-\frac{\partial H}{\partial x_2} = \dot{p}_2 = -p_1 - p_2 u$

$$\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & u \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

maximum principle

$|u^*(t)| + p_2^* x_2^* u^* \le |u(t)| + p_2^* x_2^* u(t)$

$$|u(t)| + p_2^* x_2^* u(t) = \begin{cases} [1 + p_2^* x_2^*]\, u(t) & u(t) > 0 \\ [-1 + p_2^* x_2^*]\, u(t) & u(t) < 0 \end{cases}$$

$$u^*(t) = \begin{cases} 1 & p_2^* x_2^* < -1 \\ -1 & p_2^* x_2^* > 1 \\ 0 & -1 < p_2^* x_2^* < 1 \end{cases}$$

undetermined nonnegative value $\quad p_2^* x_2^* = -1$

undetermined nonpositive value $\quad p_2^* x_2^* = 1$

1° $u^* = 1$

$$\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix}}_{M} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

$$|\lambda I - M| = \begin{bmatrix} \lambda & 1 \\ 1 & \lambda + 1 \end{bmatrix} = \lambda(\lambda + 1) - 1$$

$$\lambda^2 + \lambda - 1 = 0 \quad \Rightarrow \quad \lambda_1 = \frac{-1 - \sqrt{5}}{2} \qquad \lambda_2 = \frac{-1 + \sqrt{5}}{2}$$

$p_1 = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$

$p_2 = c_3 e^{\lambda_1 t} + c_4 e^{\lambda_2 t}$

$$\dot{x}_1 = x_2 \qquad \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \qquad |sI-M| = \begin{bmatrix}s & -1\\ -1 & s-1\end{bmatrix} = s(s-1)-1$$
$$\dot{x}_2 = x_1 + x_2$$

$$s^2 - s - 1 = 0 \Rightarrow s_1 = \frac{1-\sqrt{5}}{2} \qquad s_2 = \frac{1+\sqrt{5}}{2}$$

$$x_1 = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$
$$x_2 = K_3 e^{s_1 t} + K_4 e^{s_2 t}$$

2° $u^* = -1$

$$\begin{bmatrix}\dot{p}_1\\ \dot{p}_2\end{bmatrix} = \begin{bmatrix}0 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}p_1\\ p_2\end{bmatrix} \qquad [\lambda I - M] = \begin{bmatrix}\lambda & 1\\ 1 & \lambda-1\end{bmatrix} = \lambda(\lambda-1)-1$$

$$\lambda^2 - \lambda - 1 = 0 \Rightarrow \lambda_3 = \frac{1-\sqrt{5}}{2} \qquad \lambda_4 = \frac{1+\sqrt{5}}{2}$$

$$p_1 = c_5 e^{\lambda_3 t} + c_6 e^{\lambda_4 t}$$
$$p_2 = c_7 e^{\lambda_3 t} + c_8 e^{\lambda_4 t}$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \qquad [sI - M] = \begin{bmatrix}s & -1\\ -1 & s+1\end{bmatrix} = s(s+1)-1$$

$$s^2 + s - 1 = 0 \Rightarrow s_3 = \frac{-1-\sqrt{5}}{2} \qquad s_4 = \frac{-1+\sqrt{5}}{2}$$

$$x_1 = K_5 e^{s_3 t} + K_6 e^{s_4 t}$$
$$x_2 = K_7 e^{s_3 t} + K_8 e^{s_4 t}$$

3° $u^* = 0$

$$\begin{bmatrix}\dot{p}_1\\ \dot{p}_2\end{bmatrix} = \begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}\begin{bmatrix}p_1\\ p_2\end{bmatrix} \qquad \lambda^2 - 2 = 0 \Rightarrow \lambda_5 = -\sqrt{2} \qquad \lambda_6 = +\sqrt{2}$$

$$p_1 = c_9 e^{\lambda_5 t} + c_{10} e^{\lambda_6 t}$$
$$p_2 = c_{11} e^{\lambda_5 t} + c_{12} e^{\lambda_6 t}$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}p_1\\ p_2\end{bmatrix} \qquad s^2 - 2 = 0 \Rightarrow s_5 = -\sqrt{2} \qquad s_6 = +\sqrt{2}$$

$$x_1 = K_9 e^{s_5 t} + K_{10} e^{s_6 t}$$
$$x_2 = K_{11} e^{s_5 t} + K_{12} e^{s_6 t}$$

$${}^* = \begin{cases} 1 & \text{if } (c_3 e^{\lambda_1 t} + c_4 e^{\lambda_2 t})(K_3 e^{s_1 t} + K_4 e^{s_2 t}) < -1 \\ -1 & \text{if } (c_7 e^{\lambda_3 t} + c_8 e^{\lambda_4 t})(K_7 e^{s_3 t} + K_8 e^{s_4 t}) > 1 \\ 0 & \text{if } -1 < (c_{11} e^{\lambda_5 t} + c_{12} e^{\lambda_6 t})(K_{11} e^{s_5 t} + K_{12} e^{s_6 t}) < 1 \\ \text{undetermined nonnegative value} & p_2^* x_2^* = -1 \\ \text{undetermined nonpositive value} & p_2^* x_2^* = +1 \end{cases}$$