HW #5 7, 11, 12

HW #6 2, 3, 13

5.4. MINIMUM-TIME PROBLEMS

$$\dot{x} = a(x, u, t) \qquad |u_i(t)| \leq 1 \quad , t \in [t_0, t^*]$$

$$J(u) = \int_{t_0}^{t_f} dt = t_f - t_0$$

The objective is to transfer a system from an arbitrary initial state to a specified target set in minimum time.

Can we do it?

see:

(Ex. 5.4-1)

Logic

1) The optimal control (if it exists) is maximum effort during the entire time interval, that is $|u(t)| = 1$

2) For certain values of the initial conditions the optimal control does not exist

REACHABLE STATES from $x(t_0) = x_0$ and for $[t_0, t]$

Collection of all values $x(t)$, with $x(t_0) = x_0$, is called the set of reachable states at t.

It depends on x_0, t_0 and t.

Notation $R(t)$

(Ex. 5.4-2)

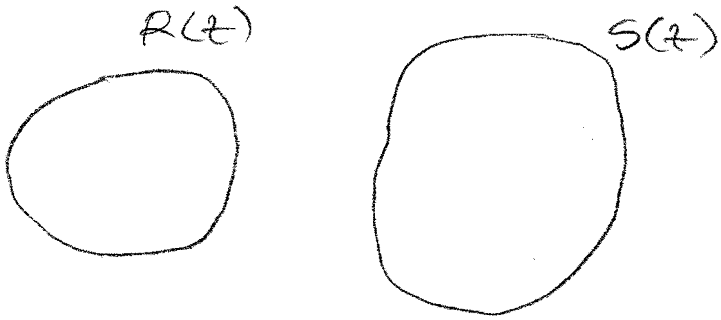
$$\dot{x} = u \qquad -1 \le u(t) \le +1$$

$$x(t) = x_0 + \int_{t_0}^{t} u(\tau)\,d\tau$$

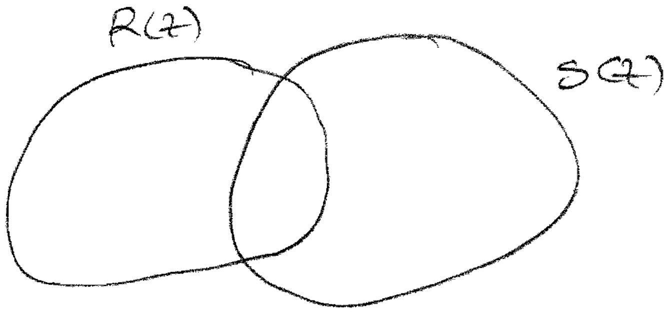
$\Rightarrow$

$$x_0 - (t - t_0) \le x(t) \le x_0 + (t - t_0)$$

$$R(t) = [\, x_0 - (t - t_0) \,,\; x_0 + (t - t_0) \,]$$

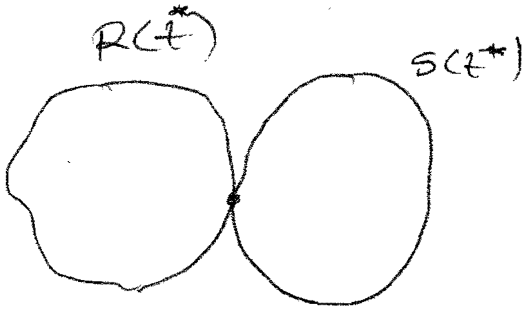


$\Rightarrow$ time-optimal control does not exist



$\Rightarrow$ time-optimal control exists

R(t) and S(t) must have at least one point in common.



! { General theorems for the existence of the solution are not available

solution at the earliest time t^* when S(t) and R(t) meet

The Form of the Optimal Control for a Class of Minimum-Time Problems

$$\dot{x} = a(x,t) + B(x,t)\,u(t)$$

$$M_{i-} \le u_i \le M_{i+}, \quad i = 1,2,\dots,m \qquad t \in [t_0, t^*]$$

The Hamiltonian is

$$H(x,u,p) = 1 + p^T \left(a(x,t) + B(x,t)\,u(t)\right)$$

From the minimum principle it is necessary that

$$1 + p^{*T}\left(a(x^*,t) + B(x^*,t)\,u^*\right) \le 1 + p^{*T}\left[a(x^*(t),t) + B(x^*(t),t)\right.$$

or

$$p^{*T} B(x^*,t)\,u^* \le p^{*T} B(x^*(t),t)\,u(t)$$

$$B = [\,b_1 \mid b_2 \mid \dots \mid b_m\,]$$

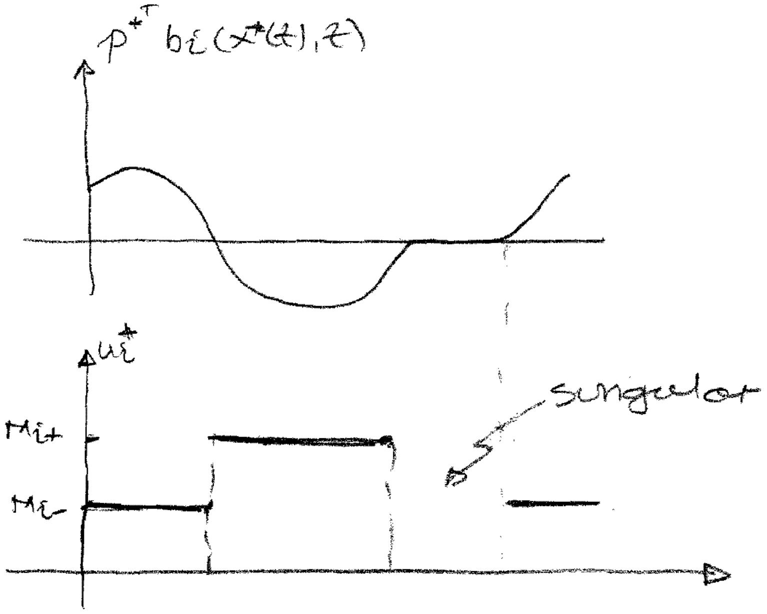
$$p^{*T} B(x^*,t)\,u(t) = \sum_{i=1}^{m} p^{*T} b_i(x^*,t)\,u_i(t)$$

Assuming that the control components are independent of one another we then must minimize $p^{*T} b_i(x^*)\,u_i$

$$u_i^*(t) = \begin{cases} M_{i+} & \text{for } p^{*T} b_i(x^*(t),t) < 0 \\ M_{i-} & \text{for } p^{*T} b_i(x^*(t),t) > 0 \\ \text{undetermined} & \text{for } p^{*T} b_i(x^*(t),t) = 0 \end{cases}$$

BANG-BANG PRINCIPLE

singular condition



TIME INVARIANT CONTROL:

$$\dot{x} = Ax + Bu \qquad |u_i(t)| < 1 \quad , i = 1, 2, \ldots, m$$

Assuming that the system is completely controllable and normal, find a control (if one exists), which transfers the system from an arbitrary initial state x_0 to the final time $x(t_f) = 0$ in minimum time.
(Pontryagin et al. 1962 The Mathematical Theory of Optimal Processes, ⇒

EXISTENCE THEOREM: All eigenvalues of A have non-positive real parts ⇒ time-optimal control exists

UNIQUENESS THEOREM: If an extremal exists, the it is unique

NUMBER OF SWITCHING:
If the eigenvalues of A are real, and a time optimal control exists, then each control component can switch at most $(n-1)$ times.

5.24. $\dot{x} = 2x + u$

$|u(t)| \le 1$

minimum time problem

$\lambda = 2 \Rightarrow$ optimal control might not exist

$H = 1 + p(2x+u)$

$-\frac{\partial H}{\partial x} = \dot{p} = -p \Rightarrow p = c_1 e^{-t}$

minimum principle

$1 + p_1^*(2x^* + u^*) \le 1 + p_1^*(2x^* + u)$

$p_1^* u^* \le p_1^* u$

$$u^* = \begin{cases} +1 & p_1^* < 0 \Rightarrow c_1 e^{-t} < 0 \Rightarrow c_1 < 0 \\ -1 & p_1^* > 0 \Rightarrow c_1 e^{-t} > 0 \Rightarrow c_1 > 0 \end{cases}$$

$u^* = +1$

$x(t) = x_0 e^{2(t-t_0)} + \int_{t_0}^{t} e^{2(t-\tau)} d\tau$

$x(t) = 0 = x_0 e^{2(t-t_0)} + e^{2t} \cdot \int_{t_0}^{t} e^{-2\tau} d\tau$

$e^{-2t_0} x_0 = -\int_{t_0}^{t} e^{-2\tau} d\tau = +\frac{1}{2} e^{-2\tau}\Big|_{t_0}^{t} \Rightarrow -\frac{1}{2} < x_0 \le 0$ 1° $t \to \infty$, $t_0 \to 0$; 2° $t \to t_0$

$u^* = -1$

$e^{-2t_0} x_0 = +\int_{t_0}^{t} e^{-2\tau} d\tau = -\frac{1}{2} e^{-2\tau}\Big|_{t_0}^{t} \Rightarrow 0 \le x_0 < 1/2$

1° $t \to \infty$, $t_0 \to 0$

2° $t \to t_0$

$2x_0 = -e^{-2t_f} + 1$

for $u^* = -1$ and $u^* = +1$

$\{-1/2 < x_0 < 1/2\}$

$e^{-2t_f} = 1 - 2x_0$

$-2t_f = \ln(1 - 2x_0) \Rightarrow$ $1 - 2x_0 > 0$, $x_0 < 1/2$

For these initial states we can reach the origin

$e^{-2t_0} x_0 = \frac{1}{2}(e^{-2t} - e^{-2t_0})$

$2x_0 + 1 > 0$

$\boxed{x_0 > -\frac{1}{2}}$

$2x_0 = e^{-2t} - 1 \Rightarrow -2t_f \ln e = \ln(2x_0 + 1) \Rightarrow t_f = -\frac{1}{2} \ln(2x_0 + 1)$

b) If $\boxed{a_i = \lambda \le 0}$, then an optimal control exists that transfers any initial state x_0 to the origin (Theo

$\boxed{a_i > 0}$

$\dot{x}_i = a_i x_i + b_i u$

$|u(t)| \le 1.0 \qquad b_i \ne 0$

$u^*(t) = \pm 1$

$$x_i(t) = e^{a_i(t-t_0)} x_i^0 + \int_{t_0}^{t} e^{a_i(t-\tau)} (\pm b_i)\, d\tau$$

$$0 = e^{a_i(t-t_0)} x_i^0 + e^{a_i t} (\pm b_i) \int_{t_0}^{t} e^{-a_i \tau} d\tau$$

$$e^{-a_i t_0} x_i^0 = \mp b_i \int_{t_0}^{t} e^{-a_i \tau} d\tau$$

$$e^{-a_i t_0} x_i^0 = \pm \frac{b_i}{a_i} e^{-a_i \tau} \Big|_{t_0}^{t} = \pm \frac{b_i}{a_i} \left(e^{-a_i t} - e^{-a_i t_0}\right)$$

$\boxed{u^*(t) = +1}$

$$e^{-a_i t_0} x_i^0 = \frac{b_i}{a_i} \left(e^{-a_i t} - e^{-a_i t_0}\right)$$

$$\left.\begin{array}{ll} 1^\circ\ t \to \infty & t_0 \to 0 \\ 2^\circ\ t \to t_0 & \end{array}\right\} \Rightarrow -\frac{|b_i|}{a_i} < x_i^0 \le 0$$

$\boxed{u^*(t) = -1}$

$$e^{-a_i t_0} x_i^0 = -\frac{b_i}{a_i} \left(e^{-a_i t} - e^{-a_i t_0}\right)$$

$$\left.\begin{array}{ll} 1^\circ\ t \to \infty & t_0 \to 0 \\ 2^\circ\ t \to t_0 & \end{array}\right\} \Rightarrow 0 \le x_i^0 < \frac{|b_i|}{a_i}$$

for $u^*(t) = +1$ and $u^*(t) = -1$

$$-\frac{|b_i|}{a_i} < x_i^0 < \frac{|b_i|}{a_i}$$

For these initial states we can reach the origin.

If $|x_i^0| \ge \frac{|b_i|}{a_i}$ an optimal control doesn't exist.