5.3 PONTRYAGIN'S MINIMUM PRINCIPLE AND STATE INEQUALITY CONSTRAINTS

$$J(u) - J(u^*) = \Delta J \geq 0$$

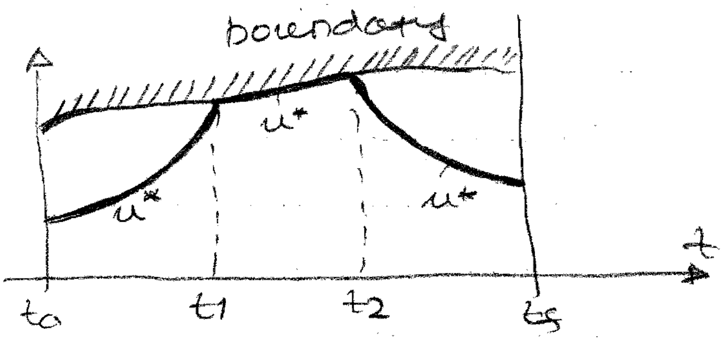
$$\Delta J(u^*, \delta u) = \underbrace{\delta J(u^*, \delta u)}_{\text{linear in } \delta u} + \overbrace{h.o.t.}^{\text{tend to zero as } \delta u \to 0}$$

The necessary condition for u^* to be an extremal control is that the variation $\delta J(u^*, \delta u) = 0$ for all admissible controls.

If $\quad |u(t)| \leq 1 \quad , \qquad t \in [t_0, t_f]$

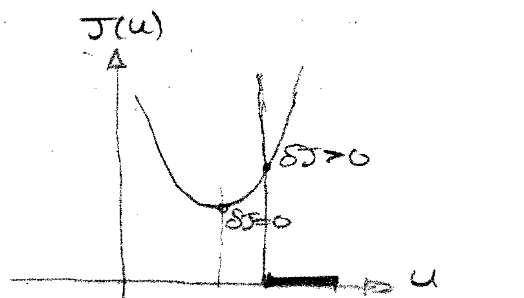
then δu is arbitrary only if the extremal control is strictly within the boundary for all time in the interval $[t_0, t_f]$.

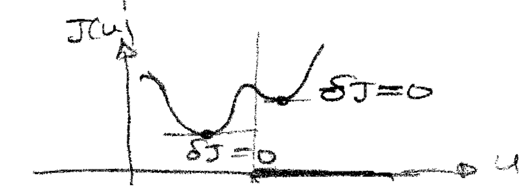
If the extremal control lies on a boundary during at least one subinterval $[t_1, t_2] \subset [t_0, t_f]$ then



$t \in [t_1, t_2]$

$\Rightarrow \quad \delta J(u^*, \delta u) \geq 0$





$H = g + p\,a$ | $\frac{\partial H}{\partial x} = -\dot{p}$ | $\frac{\partial H}{\partial u} = 0$

$\dot{x} = a$ | $\frac{\partial H}{\partial p} = \dot{x}$

$J = \int g\,dt$

CONCLUSION:
$$\begin{cases} \delta J(u^*, \delta u) \geq 0 & \text{on the boundary} \\ \delta J(u^*, \delta u) = 0 & \text{inside the boundary} \end{cases}$$

(Eqs. 5.1-9 and 5.1-13 from Sec 5.1)

$$\Delta J(u^*, \delta u) = \left[\frac{\partial h}{\partial x}(x^*(t_f), t_f) - p^*(t_f)\right]^T \delta x_f +$$

$$+ \left[H(x^*(t_f), u^*(t_f), p^*(t_f), t_f) + \frac{\partial h}{\partial t}(x^*(t_f), t_f)\right]\delta t_f$$

$$+ \int_{t_0}^{t_f} \Big\{ \left[\dot{p}^* + \frac{\partial H}{\partial x}(x^*(t), u^*(t), p^*(t), t)\right]^T \delta x$$

$$+ \left[\frac{\partial H}{\partial u}(x^*(t), u^*(t), p^*(t), t)\right]^T \delta u$$

$$+ \left[\frac{\partial H}{\partial p}(x^*(t), u^*(t), p^*(t), t) - \dot{x}^*(t)\right]^T \delta p \Big\}\, dt + \text{h.o.t}$$

$$= \delta J + \text{h.o.t}$$

- $\dot{x}^* = \frac{\partial H}{\partial p} = a$ satisfied
- $\dot{p}^* = -\frac{\partial H}{\partial x}$ can be chosen arbitrary
- boundary conditions assumed to be satisfied

$$\Rightarrow \Delta J(u^*, \delta u) = \int_{t_0}^{t_f} \left[\frac{\partial H}{\partial u}(x^*(t), u^*(t), p^*(t), t)\right]^T \delta u\, dt + \text{h.o.t}$$

$$\left[\frac{\partial H}{\partial u}(x^*, u^*, p^*, t)\right]^T \cdot \delta u \doteq H(x^*, u^* + \delta u, p^*, t) - H(x^*, u^*, p^*, t)$$

$$\Rightarrow \Delta J(u^*, \delta u) = \int_{t_0}^{t_f} \left[H(x^*, u^* + \delta u, p^*, t) - H(x^*, u^*, p^*, t)\right] dt + \text{h.o.t}$$

$$\Delta J(u^*, \delta u) \geq 0$$

$$\Rightarrow H(x^*, u^* + \delta u, p^*, t) - H(x^*, u^*, p^*, t) \geq 0$$

or, for u^* to be a minimizing control it is <u>necessary</u> that

$$\boxed{H(x^*, u^*, p^*, t) \leq H(x^*, u, p^*, t)}$$ Pontryagin's minimum principle

$\Longrightarrow$

OPTIMAL CONTROL MUST MINIMIZE THE HAMILTONIAN

<u>Summary</u>:

$$\dot{x} = a(x, u, t)$$

$$J(u) = h(x(t_f), t_f) + \int_{t_0}^{t_f} g(x(t), u(t), t)\, dt$$

$$H(x, u, p, t) \triangleq g(x, u, t) + p^T a(x, u, t)$$

NECESSARY CONDITIONS FOR THE OPTIMUM FOR CONSTRAINED CONTROL

$$\dot{x} = \frac{\partial H}{\partial p} \qquad (1)$$

$$\dot{p} = -\frac{\partial H}{\partial x} \qquad (2)$$

$$H(x^*, u^*, p^*, t) \leq H(x^*, u, p^*, t) \qquad (3)$$

Terminal conditions:

$$\left(\frac{\partial h}{\partial x}(x^*(t_f), t_f) - p^*(t_f)\right)^T \delta x_f +$$

$$\left[\left(H(x^*(t_f), u^*(t_f), p^*(t_f), t_f\right) + \frac{\partial h}{\partial t}(x^*(t_f), t_f)\right] \delta t_f = 0$$

If control is not bounded $(3) \Longrightarrow$

$$\frac{\partial H}{\partial u}(x^*, p^*, u^*, t) = 0$$

$$\frac{\partial^2 H}{\partial u^2} > 0 \Longrightarrow \text{sufficient conditions}$$

Additional Necessary Conditions:

(1) If the final time is fixed and the Hamiltonian does not depend explicitly on time, then the Hamiltonian must be constant when evaluated on an extremal trajectory, that is

$$H(x^*, p^*, u^*) = c_1 \qquad \forall t \in [t_0, t_f]$$

(2) If the final time is free, and the Hamiltonian does not explicitly depend on time, then, the Hamiltonian must be identically zero when evaluated on an extremal trajectory that is

$$H(x^*, p^*, u^*) = 0 \qquad \forall t \in [t_0, t_f]$$

~~Proof~~

Problem 5.5

(2) Show that

$$\frac{dH}{dt} = 0 \implies H(t) = \text{const}, \qquad \forall t \in [t_0, t_f]$$

since

$$H(x(t_f), u(t_f), p(t_f)) - \cancel{\frac{\partial h}{\partial t}}^{\,0} = 0 \implies H(t) = 0$$

(1) $p(t_f) = \frac{\partial h}{\partial x}(x(t_f)) = \text{const}$ or similar $x(t_f), p(t_f) = \text{const}$
$u(t_f) = \text{const}$

$$H = g + p^T a$$

$$H(t_f) = g(t_f) + p^T(t_f)\, a(t_f) = \text{const}$$

State Variable Inequality Constraints

$f(x,t) \geq 0 \qquad f_x,\ f_{xx}$ exist $\qquad \dim f = \ell$

$$\dot{x}_{n+1} = (f_1)^2 s(-f_1) + (f_2)^2 s(-f_2) + \dots + (f_\ell)^2 s(-f_\ell) = a_{n+1}$$

$$s(-f_i) = \begin{cases} 0 & f_i \geq 0 \\ 1 & f_i < 0 \end{cases}$$

Heaviside step function

$\dot{x}_{n+1} \geq 0 \quad \forall t$

$\dot{x}_{n+1} = 0$ when all constraints are satisfied

$$\int_{t_0}^{t} \dot{x}_{n+1}(t)\,dt = x_{n+1}(t) - x_{n+1}(t_0)$$

Ask for $x_{n+1}(t_0) = 0$ and $x_{n+1}(t_f) = 0$

$\Rightarrow \quad \dot{x}_{n+1}(t) = 0 \quad \forall t \in [t_0, t_f]$ (or if all constraints are satisfied)

$$\begin{cases} \dot{x} = a(x,u) \\ J = \int_{t_0}^{t_f} g(x,u,t)\,dt + h(x(t_f),t_f) \\ f(x,t) \geq 0 \end{cases}$$

Form the Hamiltonian:

$$\boxed{\begin{aligned} H(x,u,p,t) &= g(x,u,t) + p_1 a_1 + \dots + p_n a_n \\ &\quad + p_{n+1}\{f_1^2 s(-f_1) + \dots + (f_\ell)^2 s(-f_\ell)\} \end{aligned}}$$

Necessary conditions:

$$\left.\begin{aligned} \dot{x}_1 &= a_1 \\ &\vdots \\ \dot{x}_n &= a_n \\ \dot{x}_{n+1} &= a_{n+1} \end{aligned}\right\} \qquad \begin{aligned} x_1(t_0) &= 0 \\ &\vdots \\ x_{n+1}(t_0) &= 0 \end{aligned}$$

$$\dot{p}_1 = -\frac{\partial H}{\partial x_1}$$

$$\vdots$$

$$\dot{p}_n = -\frac{\partial H}{\partial x_{n+1}} = 0 \qquad \text{no } x_{n+1} \text{ in } H$$

$$H(x^*, u^*, p^*, t) \leq H^*(x^*, u, p^*, t)$$