(see pp 236)

5.5 x_0, x_f fixed $\quad \dot{x} = a(x,u)$

$$J = \int_{t_0}^{t_f} g(x,u)\,dt$$

t_f free

$$H = g(x,u) + p^T a(x,u)$$

terminal condition $\Rightarrow$

$$H(x(t_f), u(t_f), p(t_f)) - \underbrace{\cancel{\frac{\partial h}{\partial t}}}_{0} = 0$$

$$\boxed{H(x(t_f), u(t_f), p(t_f)) = 0}$$

$$\frac{dH}{dt} = \left(\frac{\partial g}{\partial x}\right)^T \dot{x} + \left(\frac{\partial g}{\partial u}\right)^T \dot{u} + \dot{p}^T a(x,u) + p^T\left(\frac{\partial a}{\partial x}\dot{x} + \frac{\partial a}{\partial u}\right.$$

From the optimality conditions know that

$$\frac{\partial H}{\partial u} = 0 = \frac{\partial g}{\partial u} + \left(\frac{\partial a}{\partial u}\right)^T p = 0$$

$$\dot{p} = -\frac{\partial H}{\partial x} = -\frac{\partial g}{\partial x} - \left(\frac{\partial a}{\partial x}\right)^T p$$

$$\frac{dH}{dt} = \cancel{\left(\frac{\partial g}{\partial x}\right)^T \dot{x}} + \left(\frac{\partial g}{\partial u}\right)\dot{u} - \cancel{\left(\frac{\partial g}{\partial x}\right) a} - \cancel{p^T\left(\frac{\partial a}{\partial x}\right) a} + \cancel{p^T \frac{\partial a}{\partial x}\dot{x}} + p^T \frac{\partial a}{\partial u}$$

$$= \left(\frac{\partial g}{\partial u}\right)\dot{u} + p^T \frac{\partial a}{\partial u}\dot{u}$$

$$= \underbrace{\cancel{\left[\frac{\partial g}{\partial u} + p^T \frac{\partial a}{\partial u}\right]}}_{0} \dot{u} = 0 \Rightarrow \frac{dH}{dt} = 0$$

$\Rightarrow H(t) = \text{const}$, but $H(t_f) = 0 \Rightarrow H(t) = 0$

$\forall t \in [t_0, t_f]$

$\dot{x} = \frac{\partial H}{\partial p}$, $\dot{p} = -\frac{\partial H}{\partial x}$, $0 = \frac{\partial H}{\partial u}$

$H = g + p^T a$ $\quad \dot{x} = a \quad J = \int_0^{t_1} g\, dt$

HW #5
5.7, 5.11, 5.12

Mar, 11, 94 (7)

(5.2) LINEAR REGULATOR

$$\dot{x} = A(t)x + B(t)u , \quad x(t_0) = x_0$$

$$J = \frac{1}{2}x^T(t_f) H\, x(t_f) + \frac{1}{2}\int_{t_0}^{t_f} [x^T Q x + u^T R u]\, dt$$

t_f fixed , $H \geq 0$, $Q \geq 0$, $R > 0$

$x(t_f)$ free , $x(t)$ and $u(t)$ are not bounded

Hamiltonian

$$H(x,u,p,t) = \frac{1}{2}x^T Q x + \frac{1}{2}u^T R u + p^T(Ax + Bu)$$

Necessary conditions for optimality are:

(1) $\dot{x}^* = A(t)x^* + B(t)u^*$, $x^*(t_0) = x_0$

(2) $\dot{p}^* = -\frac{\partial H}{\partial x} = -Qx^* - A^T p^*$, $p^*(t_f) = Hx^*(t_f)$

3) $0 = \frac{\partial H}{\partial u} = Ru^* + B^T p^* \Rightarrow u^* = -R^{-1}B^T p^*$

· Boundary conditions (pp 200 , t_f = fixed, $x(t_f)$ = free

$\Rightarrow \frac{\partial h}{\partial x}(x^*(t_f)) - p^*(t_f)) = 0$

$$\boxed{p^*(t_f) = Hx^*(t_f)}$$

eliminating $u^*(t)$ from (1)

$$\begin{bmatrix} \dot{x}^* \\ \dot{p}^* \end{bmatrix} = \begin{bmatrix} A(t) & -\overbrace{B(t)R^{-1}(t)B^T(t)}^{S(t)} \\ -Q(t) & -A^T(t) \end{bmatrix} \begin{bmatrix} x^*(t) \\ p^*(t) \end{bmatrix}, \quad \begin{bmatrix} x^*(t_0) \\ p^*(t_f) \end{bmatrix} = \begin{bmatrix} x_0 \\ Hx^*(t_f) \end{bmatrix}$$

$$\begin{bmatrix} x(t_f) \\ p(t_f) \end{bmatrix} = \overbrace{\begin{bmatrix} \phi_{11}(t_f,t) & \phi_{12}(t_f,t) \\ \phi_{21}(t_f,t) & \phi_{22}(t_f,t) \end{bmatrix}}^{\phi(t_f,t)} \begin{bmatrix} x(t) \\ p(t) \end{bmatrix}$$

$$x(t_f) = \phi_{11}(t_f,t)\, x(t) + \phi_{12}(t_f,t)\, p(t)$$
$$p(t_f) = \phi_{21}(t_f,t)\, x(t) + \phi_{22}(t_f,t)\, p(t)$$

$$p(t_f) = H x(t_f) = \phi_{21}(t_f,t)\, x(t) + \phi_{22}(t_f,t)\, p(t)$$
$$= H x(t_f) = H \phi_{11}(t_f,t) x(t) + H \phi_{12}(t_f,t)\, p(t)$$

$$\Longrightarrow$$
$$p^*(t) = [\underbrace{\phi_{22}(t_f,t) - H\varphi_{12}(t_f,t)}]^{-1} [H\varphi_{11}(t_f,t) - \varphi_{21}(t_f,t)]\, x^*(t)$$

this inverse allways exists $\forall t \in [t_0, t_f]$

(Kalman, (1960]

Contributions to the Theory of Optimal Control

Bol. Soc. Mat. Mex, 1960, 102-119.

$$p^*(t) = K(t)\, x^*(t)$$

$$\Longrightarrow \quad u^*(t) = -R^{-1}(t) B^T(t) K(t) x^*(t) \triangleq F(t)\, x^*(t)$$

thus, the optimal control law is linear

(Problem 5.9) $\quad p = Kx \Rightarrow \dot{p} = \dot{K}x + K\dot{x}$

a) $\dot{x} = Ax - Sp$

$$\dot{p} = -Qx - A^T p = \dot{K}x + K\dot{x} = \dot{K}x + KAx - KSp$$

$$-Qx - A^T K x = \dot{K}x + KAx - KSKx$$

$$(\dot{K} + KA + A^T K + Q - KSK)\, x = 0$$

$$\Rightarrow \boxed{-\dot{K} = KA + A^T K + Q - KSK} \qquad p(t_f) = \cancel{H_{tf}} x(t_f) = H x(t_f) \Rightarrow \boxed{K(t_f) = H}$$

b) K is symmetric $\Rightarrow$ $(n+1)/2$ diff. eqs

c) $x(t_f)=0 \quad \Rightarrow \quad \boxed{K(t_f)=0}$?

$\Updownarrow \; p(t_f) = H\, x(t_f) = 0$

TIME INVARIANT REGULATOR PROBLEM:

$$J = \int_0^\infty (\quad)\,dt, \qquad A, B, Q, R = \text{const}$$

$\Rightarrow$ Algebraic Riccati equation

$$0 = KA + A^T K + Q - KSK \qquad \dot{K} = 0$$

Unique positive semidefinite solution exists if (A, B) is stabilizable and $(A, \sqrt{Q})$ is detectable

$\Rightarrow$ $(A - BF)$ is stable, where $F = R^{-1} B^T K$

LINEAR TRACKING PROBLEMS

$$\dot{x} = A(t)\,x(t) + B(t)\,u(t)$$

$$J = \tfrac{1}{2}\left(x(t_f) - r(t_f)\right)^T H \left(x(t_f) - r(t_f)\right) + \tfrac{1}{2}\int_{t_0}^{t_f} \left[(x-r)^T Q (x-r) + u^T R u\right] dt$$

$$H = \tfrac{1}{2}\left[(x-r)^T Q (x-r) + u^T R u\right] + p^T (Ax + Bu)$$

$x^T Q x - r^T Q x - x Q r + r^T Q r$

$$\dot{p} = -\frac{\partial H}{\partial x} = -Qx - A^T p + Qr$$

$$0 = \frac{\partial H}{\partial u} = Ru + B^T p \Rightarrow \quad u = -R^{-1} B^T p$$

$$\begin{bmatrix}\dot{x}\\ \dot{p}\end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}\begin{bmatrix} x \\ p\end{bmatrix} + \begin{bmatrix} 0 \\ Q(t)r(t)\end{bmatrix}$$

$$\begin{bmatrix}x(t_f)\\ p(t_f)\end{bmatrix} = \Phi(t_f,t)\begin{bmatrix}x(t)\\ p(t)\end{bmatrix} + \int_t^{t_f}\Phi(t_f,\tau)\begin{bmatrix}0\\ Q(\tau)r(\tau)\end{bmatrix}d\tau$$

$$(3)\begin{cases} x(t_f) = \Phi_{11}(t_f,t)x(t) + \Phi_{12}(t_f,t)p(t) + f_1(t) \\ p(t_f) = \Phi_{21}(t_f,t)x(t) + \Phi_{22}(t_f,t)p(t) + f_2(t)\end{cases}$$

Boundary conditions

$$\frac{\partial h}{\partial x}(t_f) - p(t_f) = 0 \quad \Longrightarrow \quad p^*(t_f) = Hx^*(t_f) - r(t_f)$$

$$(3) \Longrightarrow p(t) = [\Phi_{22}(t_f,t) - H\Phi_{12}(t_f,t)]^{-1}\Big([H\Phi_{11}(t_f,t) - \Phi_{21}(t_f,t)]x(t) + [Hf_1 - Hr(t_f) - f_2(t)]\Big)$$

$$\boxed{p(t) \triangleq K(t)x(t) + s(t)}$$

$$u^*(t) = -R^{-1}B^TKx(t) - R^{-1}B^Ts(t) \triangleq F(t)x(t) + v(t)$$

$$\dot{p} = \dot{K}x + K\dot{x} + \dot{s}$$

$$\Longrightarrow \left(\dot{K}(t) + Q(t) + K(t)A(t) + A^T(t)K - K(t)B(t)R^{-1}(t)B^T(t)K(t)\right)x + \left(\dot{s} + A^T(t)s - K(t)B(t)R^{-1}(t)B^T(t)s - Q(t)\right) = 0$$

It must be satisfied for all $\underline{x(t)}$ and $\underline{r(t)}$ (they are independent)