Chapter 5

Mar. 4, 94 (6)

The Variational Approach to Optimal Control Problems

5.1 NECESSARY CONDITIONS FOR OPTIMAL CONTROL

$$\dot{x} = a(x, u, t)$$

$$J(u) = h(x(t_f), t_f) + \int_{t_0}^{t_f} g(x, u, t)\, dt$$

- admissible state and control regions are not bounded
- $x(t_0)$ and t_0 are fixed.

Assuming that h is a differentiable function we can write

$$h(x(t_f), t_f) = \int_{t_0}^{t_f} \frac{d}{dt}\left[h(x(t), t)\right] dt + h(x(t_0), t_0)$$

so that the performance measure can be expressed as

$$J(u) = \int_{t_0}^{t_f} \left\{ g(x, u, t) + \frac{d}{dt} h(x(t), t) \right\} dt + \underbrace{h(x(t_0), t_0)}_{=\text{const}}$$

no effect on optimization

thus, we consider only

$$J(u) = \int_{t_0}^{t_f} \left\{ g(x, u, t) + \frac{d}{dt} h(x(t), t) \right\} dt$$

$$= \int_{t_0}^{t_f} \left\{ g(x, u, t) + \left[\frac{\partial h}{\partial x}(x, t)\right]^T \dot{x}(t) + \frac{\partial h}{\partial t}(x, t) \right\} dt$$

Ch. 4. $\boxed{\left.\begin{array}{l} J(x) = \int_{t_0} g(x,\dot{x},t)\,dt \\ f(x_i, \dot{x}_i, t) = 0 \end{array}\right\} \Rightarrow \begin{array}{l} g_a = g + p^T f \\ \frac{\partial g_a}{\partial x_i} - \frac{d}{dt}\left[\frac{\partial g_a}{\partial \dot{x}_i}\right] = 0 \end{array}}$

Augmented functional

$$\boxed{J_a(u) = \int_{t_0}^{t_f} \left\{ g + \left(\frac{\partial h}{\partial x}\right)^T \dot{x} + \frac{\partial h}{\partial t} + p^T (a(x,u,t) - \dot{x}) \right\} dt}$$

thus

$$g_a(x,\dot{x},u,p,t) \triangleq g(x,u,t) + p^T (a(x,u,t) - \dot{x}) + \left(\frac{\partial h}{\partial x}\right)^T \dot{x} + \frac{\partial h}{\partial t}$$

so that

$$J_a(u) = \int_{t_0}^{t_f} g_a(x,\dot{x},u,p,t)\,dt$$

we introduce the variations in $\delta x, \delta \dot{x}, \delta u, \delta p, \delta t_f$

(Problem 4 $\Rightarrow$)

$$\boxed{\begin{array}{l} (1)\quad \frac{\partial g}{\partial x} - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}}\right] = 0 \Big|_{\text{at } t \in [t_0, t_f]} \\ (2)\quad \left[\frac{\partial g}{\partial \dot{x}}\right]^T \delta x_f + \left\{ g - \left(\frac{\partial g}{\partial \dot{x}}\right)^T \dot{x} \right\} \delta t_f = 0 \Big|_{t=t_f} \end{array}}$$

Applied here $\Rightarrow$

$$\delta J_a(u) = 0 = \left[\frac{\partial g_a}{\partial \dot{x}}\right]^T_{t=t_f} \delta x_f + \left\{ g_a - \left(\frac{\partial g_a}{\partial \dot{x}}\right)^T \dot{x} \right\}\Big|_{t=t_f} \delta t_f$$

$$+ \int_{t_0}^{t_f} \left\{ \left[\left(\frac{\partial g_a}{\partial x}\right)^T - \frac{d}{dt}\left(\frac{\partial g_a}{\partial \dot{x}}\right)^T\right] \delta x + \left[\left(\frac{\partial g_a}{\partial u}\right)^T - 0\right] \delta u + \left[\left(\frac{\partial g_a}{\partial p}\right)^T - 0\right] \delta p \right\} dt$$

<u>only h function</u>

$$\frac{\partial}{\partial x}\left[\left(\frac{\partial h}{\partial x}\right)^T \dot{x} + \frac{\partial h}{\partial t}\right] - \frac{d}{dt}\left\{\frac{\partial}{\partial \dot{x}}\left[\frac{\partial h}{\partial x}\right]^T \dot{x} - 0\right\}$$

$$\frac{\partial^2 h}{\partial x^2}\dot{x} + \frac{\partial^2 h}{\partial x \partial t} - \frac{d}{dt}\left(\frac{\partial h}{\partial x}\right) =$$ assuming second partial derivatives are continuous

$$\frac{\partial^2 h}{\partial x^2}\dot{x} + \cancel{\frac{\partial^2 h}{\partial x \partial t}} - \frac{\partial^2 h}{\partial x^2}\cdot\dot{x} - \cancel{\frac{\partial^2 h}{\partial x \partial t}} = 0$$

Left with:

$g_a = g + p^T(a - \dot{x})$

$\frac{\partial}{\partial x}(\ ) - \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}(\ )\right) = 0$

$$\int_{t_0}^{t_f} \left\{ \left\{ \left[\frac{\partial g}{\partial x}\right]^T - p^T \frac{\partial a}{\partial x} - \frac{d}{dt}(p^T) \right\} \delta x + \left\{ \left(\frac{\partial g}{\partial u}\right)^T + p^T \frac{\partial a}{\partial u} \right\} \delta u + \underbrace{\{ a - \dot{x} \}}_{=0} \delta p \right\} dt = 0$$

THE LAGRANGE MULTIPLIERS ARE ARBITRARY, let us take

$$\text{(II)} \quad \boxed{\dot{p} = -\left[\frac{\partial a}{\partial x}\right] p - \frac{\partial g}{\partial x}} \quad \text{COSTATE EQUATION}$$

then

$$0 = \left(\frac{\partial g}{\partial u}\right)^T + p^T \frac{\partial a}{\partial u} \quad \text{or} \quad \boxed{\left(\frac{\partial g}{\partial u}\right) + \left(\frac{\partial a}{\partial u}\right)^T p = 0}$$

$$\text{(I)} \quad \boxed{\dot{x} = a_{\ x(t_0) = x_0}} \quad \text{state equation}$$

TERMINAL CONDITIONS

$$0 = \left(\frac{\partial g_a}{\partial \dot{x}}\right)^T \delta x_f + \left[g_a - \left(\frac{\partial g_a}{\partial \dot{x}}\right)^T \dot{x}\right] \delta t_f = 0 \Big|_{t = t_f}$$

$\Rightarrow$

$$0 = \left[\frac{\partial h}{\partial x} - p\right]^T \delta x_f + \left[g_a - \left(\frac{\partial h}{\partial x} - p\right)^T \dot{x}\right] \delta t_f = 0 \Big|_{t = t_f}$$

$$g + \cancel{\left(\frac{\partial h}{\partial x}\right)^T \dot{x}} + \frac{\partial h}{\partial t} + p^T(a - \cancel{\dot{x}}) - \cancel{\left(\frac{\partial h}{\partial x}\right)^T \dot{x}} + \cancel{p \dot{x}}$$

$$\boxed{0 = \left(\frac{\partial h}{\partial x} - p\right)^T \delta x_f + \left[\left(g + \frac{\partial h}{\partial t} + p^T a\right)\right]_{t = t_f} \cdot \delta t_f = 0^*_{t_f}}$$

$\Rightarrow$ need more: n or $n+1$ equations for end point conditions and/or t_f

GENERAL OPTIMIZATION PROBLEM

$$J = \int_{t_0}^{t} g(x,u,t)\,dt + h(x(t_f))$$
$$\dot{x} = a(x,u,t),\quad x(t_0)=x_0$$

DEFINE HAMILTONIAN for

$$H(x,u,p,t) \triangleq g(x,u,t) + p^T \cdot a(x,u,t)$$

necessary conditions

$$\boxed{\begin{aligned}
&(1)\quad \dot{x}^* = \frac{\partial H}{\partial p}^* = a(x,u,t)\\
&(2)\quad \dot{p}^* = -\frac{\partial H}{\partial x}^* = -\frac{\partial a}{\partial x}(x,u,t)\cdot p - \frac{\partial g(x,u,t)}{\partial x}\\
&(3)\quad 0 = \frac{\partial H}{\partial u}^* = \frac{\partial g}{\partial u} + \left(\frac{\partial a}{\partial u}\right)^T p
\end{aligned}}$$

and the terminal conditions:

$$\boxed{\left[\frac{\partial h}{\partial x}(x^*(t_f),t_f) - p^*(t_f)\right]^T \delta x_f + \left[H(x^*(t_f),u^*(t_f),p^*(t_f),t_f) + \frac{\partial h}{\partial t}(x^*(t_f),t_f)\right]\delta t_f = 0}$$

BOUNDARY CONDITIONS:

FIXED FINAL TIME, but $x(t_f) = \begin{cases} \text{a) specified} \\ \text{b) free} \\ \text{c) lie on some surface:} \end{cases}$

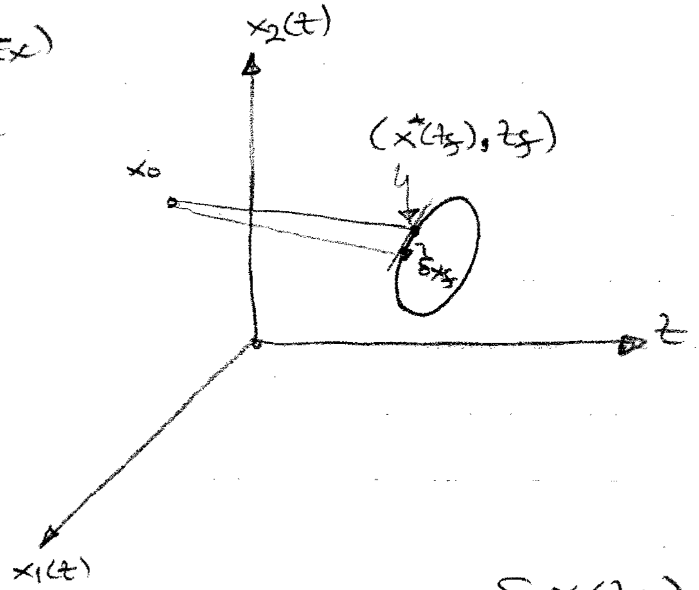
CASE I. $\underbrace{x(t_f)=x_f}_{n}$ $\quad \delta x_f = 0,\ \delta t_f = 0$ $\quad \underbrace{x(t_0)=x_0}_{n}$

CASE II. $\delta t_f = 0,\quad \delta x_f \neq 0\quad x(t_f)$ is free

$$\frac{\partial h}{\partial x}(x^*(t_f),t_f) - p^*(t_f) = 0 \qquad (n \text{ eqs})$$

CASE III. Final state lying on the surface defined by $m(x(t)) = 0$

(Ex)



$$m(x(t)) = (x_1-3)^2+(x_2-4)^2-4=0$$

$$\frac{\partial m}{\partial x} = \begin{bmatrix} \frac{\partial m}{\partial x_1} \\ \frac{\partial m}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2(x_1-3) \\ 2(x_2-4) \end{bmatrix}$$

gradient vector

tangent is normal to gradient vector $\Rightarrow$

$\delta x(t_f)$ = must be normal to the gradient

Note $\delta x_f = \delta x(t_f)$ since t_f = fixed

$$0 = \left(\frac{\partial m}{\partial x}\right)^T\Big|_{t=t_f} \delta x(t_f) = 2(x_1(t_f)-3)\,\delta x_1(t_f) + 2(x_2(t_f)-4)\,\delta x_2(t_f)$$

$\Rightarrow$

$$\delta x_2(t_f) = -\frac{(x_1-3)}{(x_2-4)}\Big|_{t_f} \delta x_1(t_f)$$

then

$$\left[\frac{\partial h}{\partial x}(x^*(t_f), t_f) - p^*(t_f)\right]^T \delta x_f$$

$$= \left(\frac{\partial h}{\partial x}(x^*(t_f)) - p^*(t_f)\right)^T \begin{pmatrix} 1 \\ -\frac{(x_1-3)}{(x_2-4)} \end{pmatrix}\Bigg|_{t=t_f} = 0$$

$$m(x^*(t_f)) = (x_1^*(t_f)-3)^2 + (x_2^*(t_f)-4)^2 - 4 = 0$$

$\Rightarrow$ 2 terminal conditions

In general: n-state variables, k constraints at t_f

$$m(x(t)) = \begin{bmatrix} m_1(x(t)) \\ m_2(x(t)) \\ \vdots \\ m_k(x(t)) \end{bmatrix} = 0$$

the final state lies on the intersection of these k-hypersurfaces at the point $(x(t_f), t_f)$

$\Rightarrow$ $\delta x(t_f)$ is normal to each of the gradient vectors

$$\left(\frac{\partial m_1}{\partial x}, \ldots, \frac{\partial m_k}{\partial x} \right)$$

$$\underbrace{\left[\frac{\partial h}{\partial x}(x^*(t_f)) - p^*(t_f) \right]^T \delta x(t_f)}_{} \triangleq v^T \delta x(t_f) = 0$$

LEMMA: this equation is satisfied iff and only if v is a linear combination of a gradient vector

$$\frac{\partial h}{\partial x}(x^*(t_f)) - p^*(t_f)) = d_1 \frac{\partial m_1}{\partial x} + d_2 \frac{\partial m_2}{\partial x} + \ldots + d_k \frac{\partial m_k}{\partial x}$$

(previous example)

$$\frac{\partial h}{\partial x} - p = d \begin{bmatrix} 2(x_1 - 3) \\ 2(x_2 - 4) \end{bmatrix} \Rightarrow 2 \text{ eqs}$$

$$m(x(t_f)) = (x_1(t_f) - 3)^2 + (x_2(t_f) - 4)^2 - 4 = 0 \quad 1 \text{ eq.}$$

$\Rightarrow$ 2 terminal conditions and d

(EXAMPLE) 5.1.1

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2 + u \\ J(u) &= \int_{t_0}^{t_f} \tfrac{1}{2} u^2 dt \end{aligned}\right\} \Rightarrow H = \tfrac{1}{2}u^2 + p_1 x_2 + p_2(-x_2+u)$$

optimality conditions

$$-\frac{\partial H}{\partial x_1} = \dot{p}_1^* = 0 \qquad (1)$$

$$-\frac{\partial H}{\partial x_2} = \dot{p}_2^* = -p_1^* + p_2^* \qquad (2)$$

$$\frac{\partial H}{\partial u} = 0 = u^* + p_2^* \qquad (3)$$

$$\dot{x}_1^* = x_2^* \qquad (4)$$

$$\dot{x}_2^* = -x_2^* - p_2^* \qquad (5)$$

$$\Rightarrow x_1^*(t) = c_1 + c_2(1-e^{-t}) + c_3(-t - \tfrac{1}{2}e^{-t} + \tfrac{1}{2}e^{t})$$

$$x_2^*(t) = c_2 e^{-t} + c_3(-1 + \tfrac{1}{2}e^{-t} + \tfrac{1}{2}e^{t}) + c_4(\tfrac{1}{2}e^{-t} - \tfrac{1}{2}e^{t})$$

$$p_1^*(t) = c_3$$

$$p_2^*(t) = c_3(1-e^{t}) + c_4 e^{t}$$

a) $t_0 = 0$, $t_f = 2$, $x(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $x(2) = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$

$\Rightarrow c_1, c_2, c_3, c_4$ easily

b) $x(0) = 0$, $t_f = 2$, $x(2)$ unspecified

$$J(u) = \frac{1}{2}\int_0^2 u^2(t)\,dt + \frac{1}{2}(x_1(2)-5)^2 + \frac{1}{2}(x_2(2)-2)^2$$

$$p(t_f) = \frac{\partial h}{\partial x}(x(t_f)) \qquad p_1(2) = x_1(2) - 5$$

$$p_2(2) = x_2(2) - 2$$

$\Rightarrow c_1, c_2, c_3, c_4$ easily

$t_f = 2$

c) $x(0) = 0,$ $m(x(t)) = 0, \Rightarrow \frac{\partial m}{\partial x} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

$J(u) = \frac{1}{2}\int_0^{t_f} u^2 dt$ $x_1(t) + 5x_2(t) = 15$

$$\begin{bmatrix} 0 - p_1(2) \\ 0 - p_2(2) \end{bmatrix} = d \cdot \begin{bmatrix} 1 \\ 5 \end{bmatrix} \Rightarrow \left. \begin{array}{l} x_1(2) + 5x_2(2) = 15 \\ p_1(2) = d = \frac{p_2(2)}{5} \end{array} \right\} \Rightarrow 3 \text{ eqs}$$

2 term + d

Problem 3.6

FREE FINAL TIME

CASE I. $x(t_f) = \text{fixed} \quad \Rightarrow \quad \delta x = 0$

$$H(x^*(t_f), u^*(t_f), p^*(t_f), t_f) + \frac{\partial h}{\partial t}(x^*(t_f), t_f) = 0 \qquad (1 \text{ eqs}) \Rightarrow t_f$$

CASE II. δx_f and δt_f are arbitrary and independent

$\Rightarrow$

$$\begin{cases} \dfrac{\partial h}{\partial x}(x^*(t_f), t_f) = p^*(t_f) & (n \text{ eqs}) \\ H(x^*(t_f), u^*(t_f), p^*(t_f), t_f) + \dfrac{\partial h}{\partial t}(x^*(t_f), t_f) = 0 & (1 \text{ eq}) \end{cases}$$

CASE III

$x(t_f)$ lies on the moving point $\theta(t)$

$$\delta x_f \doteq \left[\frac{d\theta(t_f)}{dt}\right] \delta t_f \qquad \text{(pp 140 problem 4)}$$

$$\Rightarrow \quad H + \frac{\partial h}{\partial t} + \left(\frac{\partial h}{\partial x} - p\right)^T \frac{d\theta}{dt} = 0 \Big|_{t_f}^{*} \qquad (1 \text{ eq})$$

also

$$x(t_f) = \theta(t_f) \qquad (n\text{-eqs})$$

CASE IV. Final state lying on the surface defined by $m(x) = 0 = \begin{bmatrix} m_1(x(t)) \\ m_k(x(t)) \end{bmatrix} = 0$

(similar to case III with t_f fixed)

δx_f and δt_f are independent

$$\Rightarrow H + \frac{\partial h}{\partial t} = 0_{t_f}^{*} \qquad (1 \text{ eq}) \qquad\qquad x(t_0) = x_0 \quad (n\text{-eqs})$$

and

$$\left(\frac{\partial h}{\partial x} - p\right) \delta x(t_f) = 0_{t_f}^{*} \qquad (n\text{-eqs})$$

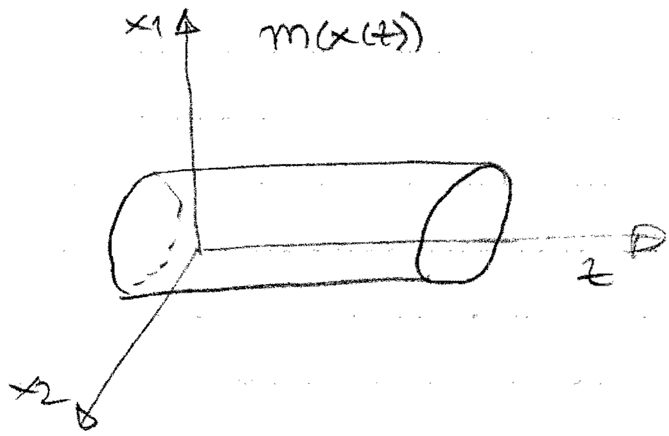
$$\Rightarrow$$

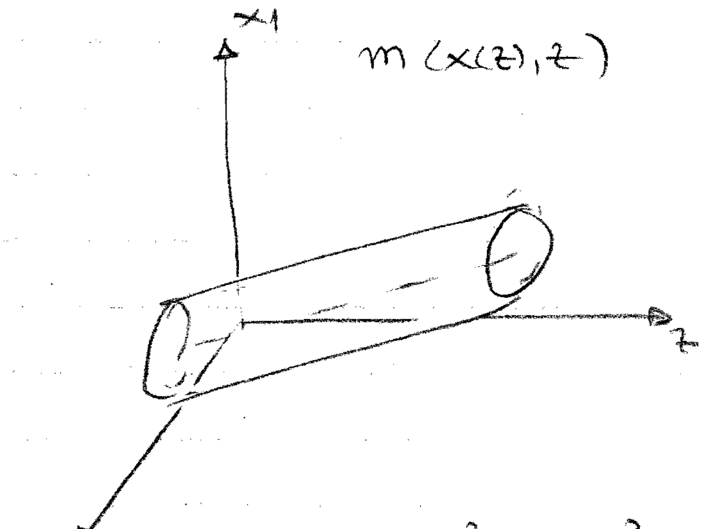
$$-\frac{\partial R}{\partial x} - p = d_1 \left(\frac{\partial m_1}{\partial x}\right) + \cdots + d_k \left(\frac{\partial m_k}{\partial x}\right)$$

CASE (V.) Final state lying on the moving surface defined by $m(x(t),t) = 0$

$\Rightarrow$ the values of δx_f depends on δt_f

$$\boxed{\begin{aligned} x(t) &= x^*(t) + \delta x(t) \\ \delta x_f &\triangleq \delta x(t_f) + \dot{x}^*(t_f)\,\delta t_f \end{aligned}}$$





$m(x(t)) = (x_1 - 3)^2 + (x_2 - 4)^2 - 4 = 0$ $\qquad$ $m(x(t),t) = (x_1-3)^2 + (x_2-4-t)^2 - 4 =$

form the vector $\begin{pmatrix} \delta x_f \\ \delta t_f \end{pmatrix}$ so that

$$\left(\frac{\partial R}{\partial x} - p\right)^T \delta x_f + \left(H + \frac{\partial R}{\partial t}\right)\delta t_f = 0$$

$$= \begin{bmatrix} \frac{\partial R}{\partial x} - p \\ H + \frac{\partial R}{\partial t} \end{bmatrix}^T \begin{bmatrix} \delta x_f \\ \delta t_f \end{bmatrix} = 0 = v^T \begin{bmatrix} \delta x_f \\ \delta t_f \end{bmatrix}$$

v must be a linear combination of the gradient vectors that is

$$\nu = d_1 \begin{bmatrix} \frac{\partial m_1}{\partial x} \\ \frac{\partial m_1}{\partial t} \end{bmatrix} + \cdots + d_k \begin{bmatrix} \frac{\partial m_k}{\partial x} \\ \frac{\partial m_k}{\partial t} \end{bmatrix} =$$

$\Longrightarrow$

$$\frac{\partial h}{\partial x} - p = d_1 \left(\frac{\partial m_1}{\partial x} \right) + \cdots + d_k \left(\frac{\partial m_k}{\partial x} \right) \quad (n \text{ eqs})$$

$$H + \frac{\partial h}{\partial t} = d_1 \left(\frac{\partial m_1}{\partial t} \right) + \cdots + d_k \left(\frac{\partial m_k}{\partial t} \right) \quad (1 \text{ eq})$$

also

$$m(x^*(t_f), t_f) = 0 \quad (k \text{ eqs})$$

$$x(t_0) = x_0 \quad (n \text{ eqs})$$

(11)

Table 5-1 SUMMARY OF BOUNDARY CONDITIONS IN OPTIMAL CONTROL PROBLEMS

Problem	*Description*	*Substitution in Eq. (5.1-18)*	*Boundary-condition equations*	*Remarks*
t_f fixed	1. $\mathbf{x}(t_f) = \mathbf{x}_f$ specified final state	$\delta \mathbf{x}_f = \delta \mathbf{x}(t_f) = \mathbf{0}$ $\delta t_f = 0$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\mathbf{x}^*(t_f) = \mathbf{x}_f$	$2n$ equations to determine $2n$ constants of integration
	2. $\mathbf{x}(t_f)$ free	$\delta \mathbf{x}_f = \delta \mathbf{x}(t_f)$ $\delta t_f = 0$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)) - \mathbf{p}^*(t_f) = \mathbf{0}$	$2n$ equations to determine $2n$ constants of integration
	3. $\mathbf{x}(t_f)$ on the surface $\mathbf{m}(\mathbf{x}(t)) = \mathbf{0}$	$\delta \mathbf{x}_f = \delta \mathbf{x}(t_f)$ $\delta t_f = 0$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)) - \mathbf{p}^*(t_f) = \sum_{i=1}^{k} d_i\left[\frac{\partial m_i}{\partial \mathbf{x}}(\mathbf{x}^*(t_f))\right]$ $\mathbf{m}(\mathbf{x}^*(t_f)) = \mathbf{0}$	$(2n + k)$ equations to determine the $2n$ constants of integration and the variables $d_1, \ldots, d_k$
t_f free	4. $\mathbf{x}(t_f) = \mathbf{x}_f$ specified final state	$\delta \mathbf{x}_f = \mathbf{0}$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\mathbf{x}^*(t_f) = \mathbf{x}_f$ $\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f) = 0$	$(2n + 1)$ equations to determine the $2n$ constants of integration and t_f
	5. $\mathbf{x}(t_f)$ free		$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f) = \mathbf{0}$ $\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f) = 0$	$(2n + 1)$ equations to determine the $2n$ constants of integration and t_f
	6. $\mathbf{x}(t_f)$ on the moving point $\boldsymbol{\theta}(t)$	$\delta \mathbf{x}_f = \left[\frac{d\boldsymbol{\theta}}{dt}(t_f)\right]\delta t_f$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\mathbf{x}^*(t_f) = \boldsymbol{\theta}(t_f)$ $\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f)$ $+ \left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f)\right]^T \left[\frac{d\boldsymbol{\theta}}{dt}(t_f)\right] = 0$	$(2n + 1)$ equations to determine the $2n$ constants of integration and t_f
	7. $\mathbf{x}(t_f)$ on the surface $\mathbf{m}(\mathbf{x}(t)) = \mathbf{0}$		$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f) = \sum_{i=1}^{k} d_i\left[\frac{\partial m_i}{\partial \mathbf{x}}(\mathbf{x}^*(t_f))\right]$ $\mathbf{m}(\mathbf{x}^*(t_f)) = \mathbf{0}$ $\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f) = 0$	$(2n + k + 1)$ equations to determine the $2n$ constants of integration, the variables $d_1, \ldots, d_k$, and t_f
	8. $\mathbf{x}(t_f)$ on the moving surface $\mathbf{m}(\mathbf{x}(t), t) = \mathbf{0}$		$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f) = \sum_{i=1}^{k} d_i\left[\frac{\partial m_i}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f)\right]$ $\mathbf{m}(\mathbf{x}^*(t_f), t_f) = \mathbf{0}$ $\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f)$ $= \sum_{i=1}^{k} d_i\left[\frac{\partial m_i}{\partial t}(\mathbf{x}^*(t_f), t_f)\right]$	$(2n + k + 1)$ equations to determine the $2n$ constants of integration, the variables $d_1, \ldots, d_k$, and t_f.