Ch 4 20, 24, 25
due in two weeks

(4.5) CONSTRAINED EXREMA

CONSTRAINED MINIMIZATION OF FUNCTIONS: (known)

Extremize $f(y_1, y_2, -, y_n, -- y_{n+m})$ subject to

$$\left.\begin{aligned} a_1(y_1, -, y_{n+m}) &= 0 \\ a_2(y_1, -, y_{n+m}) &= 0 \\ &\vdots \\ a_n(y_1, --, y_{n+m}) &= 0 \end{aligned}\right\} n \text{ constraints}$$

Form:

$f_a(y_1 -, y_{n+m}, p_1, -, p_n) =$

$$f_a = f + p_1 a_1 + p_2 a_2 + -- + p_n a_n$$

$p_j, j = 1, 2, -, n$ = Langrange multiplier

necessary conditions for extremals

$$y^*, p^* \left\{ \begin{aligned} &\frac{\partial f_a}{\partial y_j}(y_1^*, -, y_{n+m}^*, p_1^*, -, p_n^*) = 0, \quad j = 1, 2, -, n+m \\ &a_i(y_1^*, y_2^*, -, y_{n+m}^*) = 0 = \frac{\partial f_a}{\partial p_i}, \quad i = 1, -, n \end{aligned}\right.$$

CONSTRAINED MINIMIZATION OF FUNCTIONALS

$$J(x) = \int_{t_0}^{t_f} g(x, \dot{x}, t)\, dt \qquad x \in \mathbb{R}^{n+m}$$

$f_i(x, t) = 0, \quad i = 1, -, n \qquad n\text{-constraints}$

$\Rightarrow$ only m-components are independent

LAGRANGE MULTIPLIERS

$$J_a(x) = \int_{t_0}^{t_f} [g(x, \dot{x}, t) + p^T(t) f(x(t), t)]\, dt, \qquad p \in \mathbb{R}^m$$

$$\delta J_a(x, \delta x, p, \delta p) =$$

$$= \int_{t_0}^{t_f} \left\{ \left[\frac{\partial g^T}{\partial x}(x, \dot{x}, t) + p^T \frac{\partial f}{\partial x}(x, t) \right] \delta x + \left[\frac{\partial g^T}{\partial \dot{x}}(x, \dot{x}, t) \right] \delta \dot{x} + f^T(x, t)\, \delta p \right\} dt$$

$$\delta J_a = \int_{t_0}^{t_f} \left[\left\{ \left[\frac{\partial g^T}{\partial x}(x, \dot{x}, t) + p^T \frac{\partial f}{\partial x}(x, t) - \frac{d}{dt}\left[\frac{\partial g^T}{\partial \dot{x}}(x, \dot{x}, t) \right] \right\} \delta x + \cancelto{0}{f^T(x, t)}\, \delta p \right] dt = 0$$

On an extremal trajectory

(2) $\boxed{f(x^*, t) = 0}$ n-equations

$$\Rightarrow \boxed{\frac{\partial g^T}{\partial x}((x^*, \dot{x}^*, t), t) + p^{*T} \frac{\partial f}{\partial x}(x^*, t) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}}(x^*, \dot{x}^*, t) \right] = 0}$$

If we define

$$g_a = g + p^T f$$

then

(1) $$\boxed{\frac{\partial g_a}{\partial x}(x^*, \dot{x}^*, p^*, t) - \frac{d}{dt}\left[\frac{\partial g_a}{\partial \dot{x}}(x^*, \dot{x}^*, p^*, t) \right] = 0}$$

$\underline{n+m}$ second order differential equations together with (2) $\Rightarrow$ $2n+m$ equations for x_{n+m} and p_n

DIFFERENTIAL EQUATIONS CONSTRAINTS

$$J(x) = \int_{t_0}^{t_f} g(x, \dot{x}, t)\, dt$$

$x \in \mathbb{R}^{n+m}$

only m-components of $x(t)$ are independent

(1) $\boxed{f_i(x, \dot{x}, t) = 0,}$ $\quad i = 1, 2, \ldots, n$

Form

$$J_a(x, p) = \int_{t_0}^{t_f} [g(x, \dot{x}, t) + p^T f(x, \dot{x}, t)]\, dt$$

$$\delta J_a(x, \delta x, p, \delta p) = \int_{t_0}^{t_f} \Big\{ \left(\frac{\partial g}{\partial x}^T + p^T \frac{\partial f}{\partial x} \right) \delta x +$$

$$+ \left[\left(\frac{\partial g}{\partial \dot{x}} \right)^T + p^T \frac{\partial f}{\partial \dot{x}} \right] \delta \dot{x} + f^T \cdot \delta p \Big\}\, dt$$

Integrating by parts

$$\delta J_a = \int_{t_0}^{t_f} \left(\frac{\partial g}{\partial x}^T + p^T \frac{\partial f}{\partial x} - \frac{d}{dt} \left[\frac{\partial g}{\partial \dot{x}}^T + p^T \frac{\partial f}{\partial \dot{x}} \right] \right) \delta x\, dt$$

$$+ \int_{t_0}^{t_f} f^T \underbrace{\cancel{\delta p}}_{0}\, dt$$

$$\delta J_a = 0 \Rightarrow \boxed{\frac{\partial g}{\partial x} + \left(\frac{\partial f}{\partial x} \right)^T p - \frac{d}{dt} \left(\frac{\partial g}{\partial \dot{x}} + \left(\frac{\partial f}{\partial \dot{x}} \right)^T p \right) = 0^*}$$

optimal

Augmented integrand

$$g_a = g + p^T f$$

$$\Rightarrow$$

(2) $\boxed{\frac{\partial g_a}{\partial x}(x^*, \dot{x}^*, p^*, t) - \frac{d}{dt} \left[\frac{\partial g_a}{\partial \dot{x}}(x^*, \dot{x}^*, p^*, t) \right] = 0}$

(1) & (2) $\Rightarrow$ $2n+m$ second-order diff. equations

$$\frac{\partial g_a}{\partial w_1} - \frac{d}{dt}\left(\frac{\partial g_a}{\partial \dot{w}_1}\right) = 0$$ (6a)

Ex) 4.5-5 $\dot{x}_1 = x_2 - x_1$

$\dot{x}_2 = -2x_1 - 3x_2 + u$

$J(x, u) = \frac{1}{2}\int_{t_0}^{t_f} (x_1^2 + x_2^2 + u^2)\, dt$

$x_1 = w_1$

$x_2 = w_2$

$u = w_3$

$J(w) = \frac{1}{2}\int_{t_0}^{t_f} (w_1^2 + w_2^2 + w_3^2)\, dt$

(1) $\dot{w}_1 = w_2 - w_1$

(2) $\dot{w}_2 = -2w_1 - 3w_2 + w_3$

$$g_a = \frac{1}{2}(w_1^2 + w_2^2 + w_3^2) + p_1(w_2 - w_1 - \dot{w}_1) + p_2(-2w_1 - 3w_2 + w_3 - \dot{w}_2)$$

$-\frac{d}{dt}\left(\frac{\partial g_a}{\partial \dot{w}_1}\right) = w_1 - p_1 - 2p_2 - \frac{d}{dt}(-p_1) = 0 \Rightarrow \dot{p}_1 = p_1 + 2p_2 - w_1$ (3)

$= w_2 + p_1 - 3p_2 - \frac{d}{dt}(-p_2) = 0 \Rightarrow \dot{p}_2 = -p_1 + 3p_2 - w_2$ (4)

$-\frac{d}{dt}\left(\frac{\partial g_a}{\partial \dot{w}_3}\right) = 0 = w_3 + p_2$ (5)

(1) – (5) $\Longrightarrow$ w_1, w_2, w_3, p_1, p_2

Isoperimetric Constraints: (INTEGRAL CONSTRAINTS)

$$\int_{t_0}^{t_f} e_i(x, \dot{x}, t)\,dt = \text{const} = c_i \quad , \quad i = 1, 2, \dots, r$$

$\Downarrow$

(control: total fuel or energy to perform a task = const)

Defining

$$z_i(t) \triangleq \int_{t_0}^{t} e_i(x, \dot{x}, t)\,dt \quad , \quad i = 1, 2, \dots, r$$

$$\Rightarrow \left.\begin{array}{l} \dot{z}_i(t) = e_i(x, \dot{x}, t), t) \\ z_i(t_0) = 0 \quad , \quad z_i(t_f) = c_i \end{array}\right\} \quad i = 1, 2, \dots, r$$

or,

$$\dot{z} = e(x, \dot{x}, t)$$

Augmented system

$$\boxed{g_a(x, \dot{x}, p, \dot{z}, t) \triangleq g(x, \dot{x}, t) + p^T(e(x, \dot{x}, t) - \dot{z})}$$

$\Rightarrow$

$(n+m)$ eqs. $\quad \dfrac{\partial g_a}{\partial x} - \dfrac{d}{dt}\left(\dfrac{\partial g_a}{\partial \dot{x}}\right) = 0$

and

(r) eqs. $\quad \cancel{\dfrac{\partial g_a}{\partial z}}_{0} - \dfrac{d}{dt}\left(\dfrac{\partial g_a}{\partial \dot{z}}\right) = 0 \quad \Rightarrow \dfrac{d}{dt}\left(\dfrac{\partial g_a}{\partial \dot{z}}\right) = 0 \Rightarrow \dfrac{d}{dt}(p^*) = 0$

$\Rightarrow \underline{p^* = \text{const}}$

(ONLY R-OF THEM)

In addition to

$\underline{\underline{(r)}}$ eqs. $\quad \dot{z} = e(x^*, \dot{x}^*, t) \qquad z_i^*(t_f) = c_i, \qquad z_i(t_0) = 0$

$$\frac{\partial g_a}{\partial x_1} - \frac{d}{dt}\left(\frac{\partial g_a}{\partial \dot{x}_1}\right) = 0$$

$$\frac{\partial g_a}{\partial z} - \frac{d}{dt}\left[\frac{\partial g_a}{\partial \dot{z}}\right] = 0 \quad \Rightarrow \quad \dot{p}^*(t) = 0$$

- (Exp) 4.5-7 (control problem)

$$\dot{x}_1 = -x_1 + x_2 + u$$
$$\dot{x}_2 = -2x_1 - 3x_2 + u$$

$$J(x,u) = \frac{1}{2}\int_{t_0}^{t_f} (x_1^2 + x_2^2)\,dt$$

The total energy to be expended is

$$\int_{t_0}^{t_f} u^2(t)\,dt = c$$

<u>Problem reformulation:</u>

$w_1 = x_1$
$w_2 = x_2$
$w_3 = u$

$$J(w) = \frac{1}{2}\int_{t_0}^{t_f} [w_1^2 + w_2^2]\,dt$$

$$\dot{w}_1 = -w_1 + w_2 + w_3$$
$$\dot{w}_2 = -2w_1 - 3w_2 + w_3$$

$$\int_{t_0}^{t_f} w_3^2(t)\,dt = c$$

$$g_a(w, \dot{w}, p, \dot{z}) = \frac{1}{2}(w_1^2 + w_2^2) + p_1(-w_1 + w_2 + w_3 - \dot{w}_1) + p_2(-2w_1 - 3w_2 + w_3 - \dot{w}_2) + p_3(w_3^2 - \dot{z})$$

$$\Rightarrow \begin{cases} (1) & \dot{w}_1 \overset{*}{=} -w_1 + w_2 + w_3 \\ (2) & \dot{w}_2 \overset{*}{=} -2w_1 - 3w_2 + w_3 \\ r=1\ (3) & \dot{z} \overset{*}{=} w_3^2, \quad z(t_0) = 0, \quad z(t_f) = c \end{cases}$$

$$0 \overset{*}{=} \frac{\partial g_a}{\partial w_1} = w_1 - p_1 - 2p_2 - \frac{d}{dt}(-p_1) \Rightarrow \quad \dot{p}_1 \overset{*}{=} p_1 + 2p_2 - w_1 \quad (4)$$

$$0 \overset{*}{=} \frac{\partial g_a}{\partial w_2} = w_2 + p_1 - 3p_2 - \frac{d}{dt}(-p_2) \Rightarrow \quad \dot{p}_2 \overset{*}{=} -p_1 + 3p_2 - w_2 \quad (5)$$

$$0 \overset{*}{=} \frac{\partial g_a}{\partial w_3} = p_1 + p_2 + 2p_3 w_2 - 0 \Rightarrow \quad 0 \overset{*}{=} p_1 + p_2 + 2w_3 p_3 \quad (6)$$