Feb. 25, 94 (5)

$\boxed{4.4}$ PIECEWISE SMOOTH EXTREMALS

Admissible curves were smooth (continuous and have continuous first derivatives). Now, we consider curves having piecewise-continuous first derivatives. $\dot{x}(t)$ is continuous except at a finite number of times in the interval (t_0, t_f). At a time when $\dot{x}$ is discontinuous, x is said to have a corner.

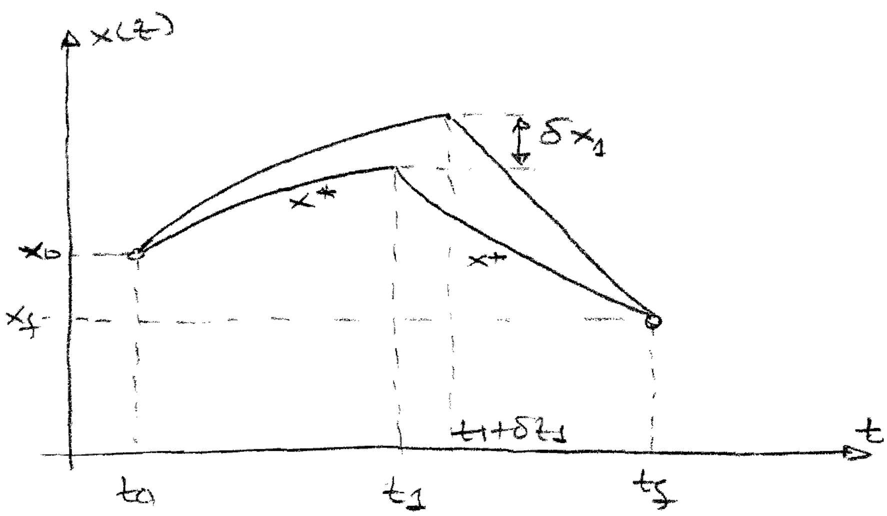
Problem:

$$\min J(x) = \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t)\,dt \qquad g \in C^2$$

$t_0, t_f, x(t_0), x(t_f)$ are fixed

$\dot{x}(t)$ has a discontinuity at some point $t_1 \in (t_0, t_f)$

$$J(x) = \int_{t_0}^{t_1} g(x, \dot{x}, t)\,dt + \int_{t_1}^{t_f} g(x, \dot{x}, t)\,dt = J_1(x) + J_2(x)$$



This is the problem 4 from previous section

(1) Euler equation (4.3-17)

$$\frac{\partial g}{\partial x}(x^*, \dot{x}^*, t) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}}(x^*, \dot{x}^*, t)\right] \stackrel{*}{=} 0^*$$

(2) boundary condition (4.3-18)

$$\left[\frac{\partial g}{\partial \dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f)\right]^T \delta x_f +$$

$$+\left\{g(x^*(t_f), \dot{x}^*(t_f), t_f) - \left[\frac{\partial g}{\partial \dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f)\right]^T \dot{x}^*(t_f)\right\} \delta t_f = 0$$

Applied to the scalar version of our problem we have two problems #4 one in forward time (F) and another one in backward time (B)

$$\delta J(x^*, \delta x) = 0 = \left[\frac{\partial g}{\partial \dot{x}}(x^*(t_1^-), \dot{x}^*(t_1^-), t_1^-)\right] \delta x_1$$

(F) $$+\left\{g(x^*(t_1^-), \dot{x}^*(t_1^-), t_1^-) - \left[\frac{\partial g}{\partial \dot{x}}(x^*(t_1^-), \dot{x}^*(t_1^-), t_1^-)\right] \dot{x}^*(t_1^-)\right\} \delta t_1$$

upper limit boundary terms ↑

$$+\int_{t_0}^{t_1} \{\underbrace{\frac{\partial g}{\partial x}(x^*(t), \dot{x}^*(t), t) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}}(x^*(t), \dot{x}^*(t), t)\right]}_{=0 = \text{Euler Eq}}\} \delta x(t)\, dt$$

↓ lower limit boundary terms

$$-\left[\frac{\partial g}{\partial \dot{x}}(x^*(t_1^+), \dot{x}^*(t_1^+), t_1^+)\right] \delta x_1$$

$$-\left\{g(x^*(t_1^+), \dot{x}^*(t_1), t_1^+) - \left[\frac{\partial g}{\partial \dot{x}}(x^*(t_1^+), \dot{x}^*(t_1^+), t_1^+)\right] \dot{x}^*(t_1^+)\right\} \delta t_1$$

(B)

$$+\int_{t_1}^{t_f} \{\underbrace{\frac{\partial g}{\partial x}(x^*, \dot{x}^*, t) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}}(x^*, \dot{x}^*, t)\right]}_{=0 = \text{Euler Eq.}}\} dt$$

Euler equation must be satisfied.

In addition, the boundary conditions

are

$$\left[\frac{\partial g}{\partial \dot{x}}(x^*(t_1), \dot{x}^*(t_1^-), t_1) - \frac{\partial g}{\partial \dot{x}}(x^*(t_1), \dot{x}^*(t_1^+), t_1)\right]\delta x_1$$

$$+\Big\{g(x^*(t_1), \dot{x}^*(t_1^-), t_1) - \left[\frac{\partial g}{\partial \dot{x}}(x^*(t_1), \dot{x}^*(t_1^-), t_1)\right]\dot{x}^*(t_1^-)$$

$$-g(x^*(t_1), \dot{x}^*(t_1^+), t_1) + \left[\frac{\partial g}{\partial \dot{x}}(x^*(t_1), \dot{x}^*(t_1^+), t_1)\right]\Big\}\delta t_1 = 0$$

$\Rightarrow$ two cases as before

1) δx_1 and δt_1 are independent

$\Rightarrow$

(4.4-5a) $$\boxed{\frac{\partial g}{\partial \dot{x}}(x^*(t_1), \dot{x}^*(t_1^-), t_1) = \frac{\partial g}{\partial \dot{x}}(x^*(t_1), \dot{x}^*(t_1^+), t_1)}$$

(4.4-5b) $$\boxed{\begin{aligned} &g(x^*(t_1), \dot{x}^*(t_1^-), t_1) - \left[\frac{\partial g}{\partial \dot{x}}(x^*(t_1), \dot{x}^*(t_1^-), t_1)\right]\dot{x}^*(t_1^-) = \\ &= g(x^*(t_1), \dot{x}^*(t_1^+), t_1) - \left[\frac{\partial g}{\partial \dot{x}}(x^*(t_1), \dot{x}^*(t_1^+), t_1)\right]\dot{x}^*(t_1^+)\end{aligned}}$$

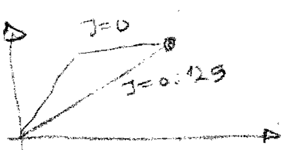
Weierstrass - Erdmann corner conditions

2) $\delta x_1 \doteq \frac{d\theta}{dt}(t_1)\,\delta t_1$ as in Problem 4

$$\left[\frac{\partial g}{\partial \dot{x}}(x^*(t_1), \dot{x}^*(t_1^-), t_1)\right]\left[\frac{d\theta}{dt}(t_1) - \dot{x}^*(t_1^-)\right] + g(x^*(t_1), \dot{x}^*(t_1^-), t_1)$$

$$= \left[\frac{\partial g}{\partial \dot{x}}(x^*(t_1), \dot{x}^*(t_1^+), t_1)\right]\left[\frac{d\theta}{dt}(t_1) - \dot{x}^*(t_1^+)\right] + g(x^*(t_1), \dot{x}^*(t_1^+), t_1)$$

- (Exp) 4.4-1 $J = 0.125$ (Problem 4 with continuous derivatives)

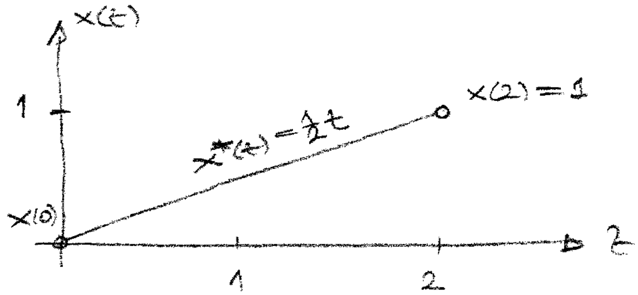


$J = 0.000$ piecewise-continuous

Example 4.4-1

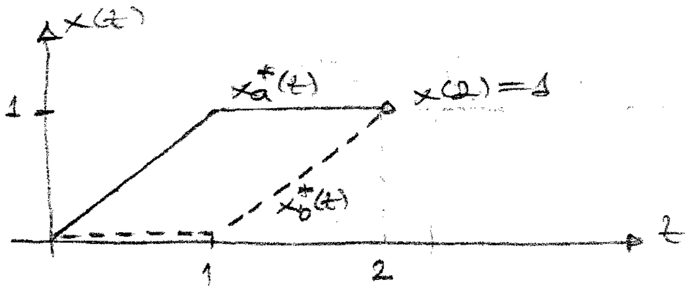
$x(0)=0, \quad x(2)=1 \qquad \min J(x) = \int_0^2 \dot{x}^2(1-\dot{x})^2 dt$

a) with continuous derivatives



$J(x^*) = \int_0^2 \left(\frac{1}{2}\right)^2\left(1-\frac{1}{2}\right)^2 dt = \frac{1}{4}\cdot\frac{1}{4}\int_0^2 dt = \frac{1}{8}$

b) with piecewise continuous derivatives



$x_a^*(t) = \begin{cases} t & 0<t<1 \\ 1 & 1<t<2 \end{cases} \Rightarrow J_a(x_a^*) = \int_0^1 1(1-1)^2 dt + \int_0^2 0\, dt = 0$

Also $\quad J_b(x_b^*) = 0$

Thus

$J(x^*) = \frac{1}{8} > J_a(x_a^*) = J_b(x_b^*)$