4.3. FUNCTIONALS OF VECTOR VARIABLES

Problem 1.

$$J(x_1, x_2, \ldots, x_n) = \int_{t_0}^{t_f} g(x_1, x_2, \ldots, x_n, \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n)\,dt$$

$x_i(t_0)$, $x_i(t_f)$, t_0, t_f are specified

$\Rightarrow$ n Euler equations

$$\frac{\partial g}{\partial x_i}(x_1^*, \ldots x_n^*, \dot{x}_1^* \ldots, \dot{x}_n^*) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}_i}(x_1^*, \ldots, x_n^*, \dot{x}_1^*, \ldots, \dot{x}_n^*)\right] = 0$$

$i = 1, 2, \ldots, n$

In the vector form: $x = (x_1, x_2, \ldots, x_n)^T$

$$J(x) = \int_{t_0}^{t_f} g(x, \dot{x}, t)\,dt \quad , x(t_0) = x_0, \quad x(t_f) = x_f$$

$$\Delta J = \int_{t_0}^{t_f} [g(x + \delta x, \dot{x} + \delta \dot{x}, t) - g(x, \dot{x}, t)]\,dt$$

$$= \int_{t_0}^{t_f} \left[\left(\frac{\partial g}{\partial x}(x, \dot{x}, t)\right)^T \delta x + \left(\frac{\partial g}{\partial \dot{x}}(x, \dot{x}, t)\right)^T \delta \dot{x} + h.o.t\right] dt$$

$$\delta J(x, \delta x) = \int_{t_0}^{t_f} \left\{\frac{\partial g}{\partial x}(x, \dot{x}, t) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}}(x, \dot{x}, t)\right]\right\}^T \delta x(t)\,dt$$

$\Rightarrow$

$$\boxed{\frac{\partial g}{\partial x}(x^*, \dot{x}^*, t) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}}(x^*, \dot{x}^*, t)\right] = 0}$$

EULER equation

Problem 4

we have seen in the previous section that

$$\delta J(x^*, \delta x) = 0 = \int_{t_0}^{t_f} \{ \overbrace{\frac{\partial g}{\partial x} - \frac{d}{dt}\left(\frac{\partial g}{\partial \dot{x}}\right)}^{EE=0} \}^T \delta x(t)\, dt$$

$$+ \left[\frac{\partial g}{\partial \dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f)\right]^T \delta x_f + g(x^*(t_f), \dot{x}^*(t_f), t_f)\,\delta t_f$$

$$- \left[\left[\frac{\partial g}{\partial \dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f)\right]^T \dot{x}^*(t_f)\right] \delta t_f$$

1) $EE = 0$ <u>since certain fixed end point problems are included in this free end point problem</u>

(2) boundary conditions

$$\boxed{\begin{aligned} 0 = & \left[\frac{\partial g}{\partial \dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f)\right]^T \delta x_f \\ & + \left\{ g(x^*(t_f), \dot{x}^*(t_f), t_f) - \left[\frac{\partial g}{\partial \dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f)\right]^T \dot{x}^*(t_f) \right\} \delta t_f = 0 \end{aligned}}$$

1) Euler equation

$$\boxed{\frac{\partial g}{\partial x}(x^*, \dot{x}^*, t) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}}(x^*, \dot{x}^*, t)\right] = 0}$$

regardless of the boundary conditions, the Euler equation must be satisfied.

(1) & (2) are necessary conditions

Integrating (1) $\Rightarrow$ $x^*(c_1, c_2, t)$

(2) $\Rightarrow$ c_1, c_2 integration constants

Table 4-1 DETERMINATION OF BOUNDARY-VALUE RELATIONSHIPS

Problem description	*Substitution*	*Boundary conditions*	*Remarks*
1. $\mathbf{x}(t_f)$, t_f both specified (*Problem 1*)	$\delta\mathbf{x}_f = \delta\mathbf{x}(t_f) = \mathbf{0}$ $\delta t_f = 0$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\mathbf{x}^*(t_f) = \mathbf{x}_f$	$2n$ equations to determine $2n$ constants of integration
2. $\mathbf{x}(t_f)$ free; t_f specified (*Problem 2*)	$\delta\mathbf{x}_f = \delta\mathbf{x}(t_f)$ $\delta t_f = 0$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f) = \mathbf{0}$	$2n$ equations to determine $2n$ constants of integration
3. t_f free; $\mathbf{x}(t_f)$ specified (*Problem 3*)	$\delta\mathbf{x}_f = \mathbf{0}$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\mathbf{x}^*(t_f) = \mathbf{x}_f$ $g(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)$ $- \left[\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)\right]^T \dot{\mathbf{x}}^*(t_f) = 0$	$(2n + 1)$ equations to determine $2n$ constants of integration and t_f
4. t_f, $\mathbf{x}(t_f)$ free and independent (*Problem 4*)	—	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f) = \mathbf{0}$ $g(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f) = 0$	$(2n + 1)$ equations to determine $2n$ constants of integration and t_f
5. t_f, $\mathbf{x}(t_f)$ free but related by $\mathbf{x}(t_f) = \boldsymbol{\theta}(t_f)$ (*Problem 4*)	$\delta\mathbf{x}_f = \frac{d\boldsymbol{\theta}}{dt}(t_f)\,\delta t_f$†	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\mathbf{x}^*(t_f) = \boldsymbol{\theta}(t_f)$ $g(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)$ $+ \left[\frac{\partial g}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), t_f)\right]^T \left[\frac{d\boldsymbol{\theta}}{dt}(t_f) - \dot{\mathbf{x}}^*(t_f)\right] = 0$†	$(2n + 1)$ equations to determine $2n$ constants of integration and t_f

† $\frac{d\boldsymbol{\theta}}{dt}$ denotes the $n \times 1$ column vector $\left[\frac{d\theta_1}{dt} \quad \frac{d\theta_2}{dt} \quad \cdots \quad \frac{d\theta_n}{dt}\right]^T$.

(Ex)

$$J(x) = \int_0^{\pi/4} \left(x_1^2 + \dot{x}_1 \dot{x}_2 + \dot{x}_2^2 \right) dt$$

boundary conditions $\quad x_1(0) = 1 \qquad x_1(\pi/4) = 2$

$\qquad\qquad\qquad\qquad\quad x_2(0) = 3/2 \qquad x_2(\frac{\pi}{4})$ free

(1) $\dfrac{\partial g}{\partial x_1} - \dfrac{d}{dt}\dfrac{\partial g}{\partial \dot{x}_1} = 0 \qquad 2x_1 - \dfrac{d}{dt}(\dot{x}_2) = 0$

(2) $\dfrac{\partial g}{\partial x_2} - \dfrac{d}{dt}\dfrac{\partial g}{\partial \dot{x}_2} = 0 \qquad 0 - \dfrac{d}{dt}(\dot{x}_1 + 2\dot{x}_2) = 0$

$$\left.\begin{array}{ll} (1) & 2x_1^* - \ddot{x}_2^* = 0 \\ (2) & \ddot{x}_1^* + 2\ddot{x}_2^* = 0 \end{array}\right\} \Rightarrow \quad \ddot{x}_1 + 4x_1 = 0$$

$$\Rightarrow x_1^*(t) = c_1 \cos 2t + c_2 \sin 2t$$

(1) $\Rightarrow \ddot{x}_2 = 2x_1 = 2c_1 \cos 2t + 2c_2 \sin 2t$

$$x_2^*(t) = -\frac{c_1}{2}\cos 2t - \frac{c_2}{2}\sin 2t + c_3 t + c_4$$

$x_1(t_f)$ specified $\Rightarrow \delta x_1(t_f) = 0 = \delta x_{1f}$

$\qquad\qquad\qquad\qquad\; \delta x_2(t_f)$ – arbitrary

$\qquad\qquad\qquad\qquad\; \delta t_f = 0$

$$\underbrace{\left[\frac{\partial g}{\partial \dot{x}_2}\left(x(\tfrac{\pi}{4}), \dot{x}(\tfrac{\pi}{4}), \tfrac{\pi}{4}\right)\right]}_{=0} \underbrace{\delta x_2(t_f)}_{\neq 0} + [\qquad]\, 0 = 0$$

$$\dot{x}_1^*(\tfrac{\pi}{4}) + 2\dot{x}_2^*(\tfrac{\pi}{4}) = 0$$

$$-2c_1 \sin(\tfrac{2\pi}{4}) + 2c_2 \cancel{\cos\tfrac{\pi}{2}}^{\,0} + 2\left(c_3 + c_1 \sin\tfrac{\pi}{2} - c_2 \cancel{\cos\tfrac{\pi}{2}}^{\,0}\right) = 2c_3$$

$$0 = 2c_3 \Rightarrow \boxed{c_3 = 0}$$

$x_1(0) = 1 = c_1 \Rightarrow \boxed{c_1 = 1}$

$x_2(0) = 3/2 = -\frac{c_1}{2} + c_4 \Rightarrow c_4 = \frac{3}{2} + \frac{c_1}{2} = 2 \quad \boxed{c_4 = 2}$

$x_1(\frac{\pi}{4}) = 2 = c_1 \cancel{\cos\frac{\pi}{2}}^{0} + c_2 \sin\frac{\pi}{2} \Rightarrow \boxed{c_2 = 2}$

thus

$$x_1^*(t) = \cos 2t + 2\sin 2t$$
$$x_2^*(t) = -\frac{1}{2}\cos 2t - \sin 2t + 2$$

sufficient conditions: /Sage pp 43/

$$J = \int_{t_0}^{t_f} g(x, \dot{x}, t)\, dt$$

$$\delta J = \int_{t_0}^{t_f} \left[\frac{\partial g}{\partial x}\delta x + \frac{\partial g}{\partial \dot{x}}\delta\dot{x}\right] dt$$

$$\delta^2 J = \frac{1}{2}\int_{t_0}^{t_f} \left\{(\delta x)^2\left[\frac{\partial^2 g}{\partial x^2} - \frac{d}{dt}\frac{\partial^2 g}{\partial \dot{x}^2}\right] + (\delta\dot{x})^2 \frac{\partial^2 g}{\partial \dot{x}^2}\right\} dt$$

$$\Delta J = \delta J + \delta^2 J + \text{h.o.t.}$$

$\delta^2 J \geq 0 \Rightarrow$ minimum