HW #3 Ch. 4,
Problems 10, 12, 17
Due Feb. 25, 1994

(4)

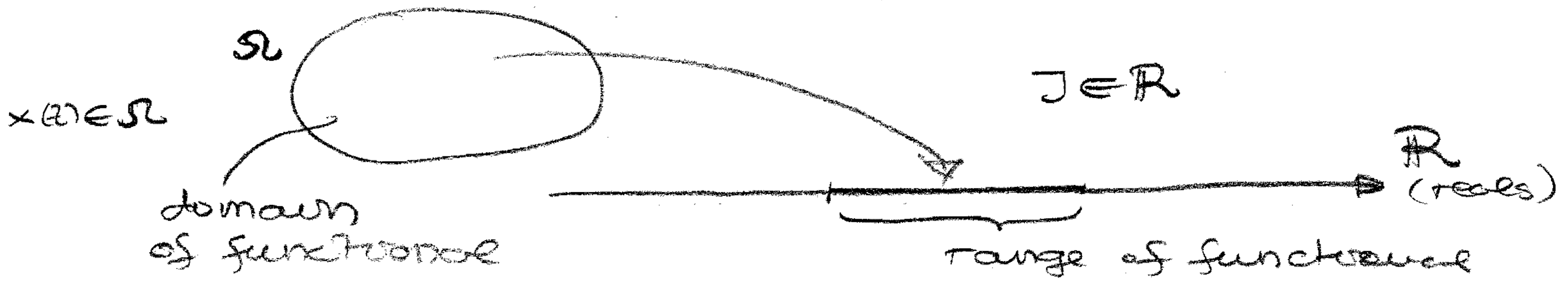
Chapter IV

4.1 Fundamental concepts of (basic definitions)

THE CALCULUS OF VARIATIONS

DEF: (4.2) FUNCTIONAL

A functional is a unique map from a set of functions in a set of reals.



(ex)

$$J(x) = \int_{t_0}^{t_f} x(t)\,dt = \text{real number}$$

DEF: (4.8) INCREMENT OF J

$$\left.\begin{array}{l} x \Rightarrow J(x) \\ x+\delta x \Rightarrow J(x+\delta x) \end{array}\right\} \Rightarrow \quad \Delta J = J(x+\delta x) - J(x)$$

δx — variation of the function x

$$\Delta J = \Delta J(x, \delta x)$$

DEF: (4.7) INCREMENT OF $f(q)$ = function of n-variables

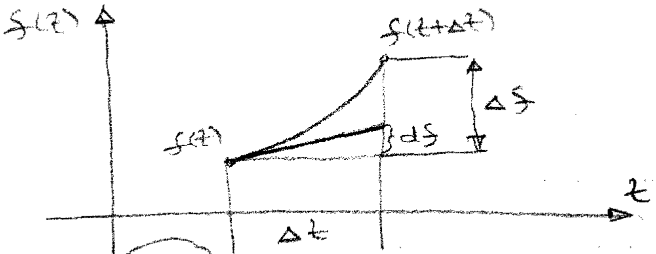
$$f(q+q_0) = f(q_0) + \frac{\partial f}{\partial q}\underbrace{(q-q_0)}_{\Delta q} + \text{R.o.t}$$

$$f(q+q_0) - f(q_0) = \Delta f(q, \Delta q) = \underbrace{\frac{\partial f}{\partial q}\Delta q}_{df(q, \Delta q)} + \text{R.o.t}$$

DEF: 4.9 f - function of n-dim vector q

$$\Delta f(q, \Delta q) = \underbrace{df(q, \Delta q)}_{\text{linear function of } \Delta q} + g(q, \Delta q)\,\|\Delta q\|$$

If $\lim_{\|\Delta q\| \to 0} \{g(q, \Delta q)\} = 0 \Rightarrow$ f is differentiable at q and df is the differential of f at point q



DEF: 4.10 The increment of a functional can be written as

$$\Delta J(x, \delta x) = \delta J(x, \delta x) + g(x, \delta x)\,\|\delta x\|$$

where

δJ is linear in δx.

If

$$\lim_{\|\delta x\| \to 0} \{g(x, \delta x)\} = 0$$

then J is said to be differentiable on x and δJ is the variation of J evaluated for the function $x(t)$.

DEF: 4-12 A functional J with domain Ω has a relative extremum at x^* if there is an $\varepsilon > 0$ such that for all functions x in Ω which satisfy

$$\|x - x^*\| < \varepsilon$$

the increment of J has the same sign

If

$$\Delta J = J(x) - J(x^*) \geq 0 \Rightarrow \text{relative minimum}$$

if

$$\Delta J = J(x) - J(x^*) \leq 0 \Rightarrow \text{" maximum}$$

FUNDAMENTAL THEOREM OF THE CALCULUS OF VARIATION:
Let x be a vector function of t in the class Ω, and $J(x)$ be a differentiable functional of x. Assume that $x(t)$ in Ω is not constrained.
IF x^* is an extremal, the variation of J must vanish on x^*, that is

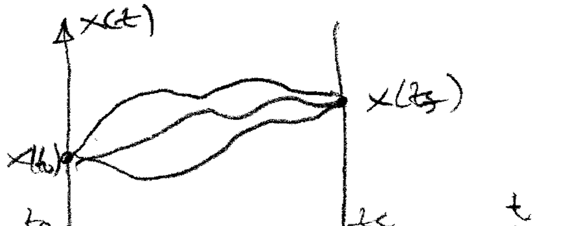
$$\boxed{\delta J(x^*, \delta x) = 0 \quad \text{for all admissible } \delta x}$$

(4.2) FUNCTIONALS OF A SINGLE FUNCTION

THE SIMPLEST VARIATIONAL PROBLEM

$$J(x) = \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t)\, dt$$

$x(t)$ is a scalar function with continuous first derivative.
$g(\cdot)$ has continuous first and second partial derivatives with respect to all of its arguments
t_0, t_f fixed, and the end points $x(t_0)$ and $x(t_f)$ are specified



$$\Delta J(x,\delta x) = J(x+\delta x) - J(x)$$

$$= \int_{t_0}^{t_f} g(x+\delta x, \dot{x}+\delta\dot{x}, t)\,dt - \int_{t_0}^{t_f} g(x,\dot{x},t)\,dt$$

$$\dot{x} = \frac{d}{dt}[x(t)] \qquad \delta\dot{x} = \frac{d}{dt}[\delta x(t)]$$

$$\Delta J(x,\delta x) = \int_{t_0}^{t_f} [g(x+\delta x, \dot{x}+\delta\dot{x}, t) - g(x,\dot{x},t)]\,dt$$

$$= \int_{t_0}^{t_f} \{\cancel{g(x,\dot{x},t)} + \frac{\partial g}{\partial x}\delta x + \frac{\partial g}{\partial \dot{x}}\delta(\dot{x}) + \text{h.o.t} - \cancel{g(x,\dot{x},t)}\}\,dt$$

$$\Rightarrow \delta J(x,\delta x) = \int_{t_0}^{t_f} \{[\frac{\partial g}{\partial x}(x,\dot{x},t)]\delta x + [\frac{\partial g}{\partial \dot{x}}(x,\dot{x},t)]\delta\dot{x}\}\,dt$$

$$\delta\dot{x} = \frac{d}{dt}[\delta x] \Rightarrow \delta x(t) = \delta x(t_0) + \int_{t_0}^{t} \delta\dot{x}(t)\,dt$$

$$\delta J(x,\delta x) = \cancel{[\frac{\partial g}{\partial \dot{x}}(x,\dot{x},t)]\delta x(t)\Big|_{t_0}^{t_f}} \rightarrow 0$$

since $\delta x(t_f) = 0$ and $\delta x(t_0) = 0$

$$+ \int_{t_0}^{t} \{\frac{\partial g}{\partial x}(x,\dot{x},t) - \frac{d}{dt}[\frac{\partial g}{\partial \dot{x}}(x,\dot{x},t)]\}\delta x(t)\,dt$$

$$\delta J(x^*,\delta x) = 0 = \int_{t_0}^{t_f} \{\frac{\partial g}{\partial x}(x^*,\dot{x}^*,t) - \frac{d}{dt}[\frac{\partial g}{\partial \dot{x}}(x,\dot{x},t)]\}\delta x(t)\,dt$$

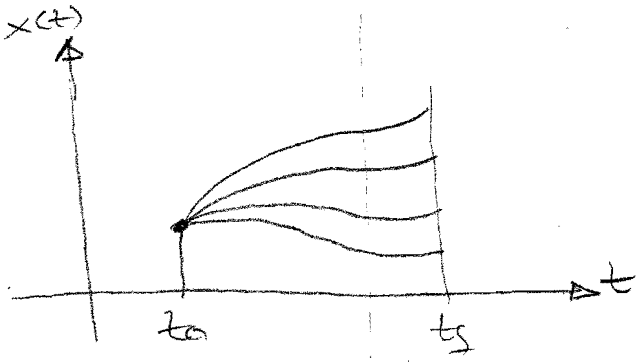
$x(t_0), x(t_f)$ GIVEN

$$\Rightarrow \boxed{\frac{\partial g}{\partial x}(x^*,\dot{x}^*,t) - \frac{d}{dt}[\frac{\partial g}{\partial \dot{x}}(x^*,\dot{x}^*,t)] = 0}$$

EULER equation [TPBVP]

nonlinear, time varying, second order [illegible] one

FINAL TIME SPECIFIED , $x(t_f)$ FREE



$\delta x(t_0) = 0$

$\delta x(t_f) \neq 0$

$$\delta J(x, \delta x) = \left[\frac{\partial g}{\partial \dot{x}}(x, \dot{x}, t)\right] \delta x(t) \Big|_{t_0}^{t_f}$$
$$+ \int_{t_0}^{t_f} \underbrace{\left\{ \left[\frac{\partial g}{\partial x}(x, \dot{x}, t)\right] - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}}(x, \dot{x}, t)\right] \right\}}_{= 0 \text{ by Euler eq.}} \delta x(t)\, dt$$

$$\Rightarrow \quad \frac{\partial g}{\partial \dot{x}}\left[x^*(t_f), \dot{x}^*(t_f), t_f\right] \underbrace{\delta x(t_f)}_{\neq 0} = 0$$

$$\Rightarrow \quad \boxed{\frac{\partial g}{\partial \dot{x}}\left(x^*(t_f), \dot{x}^*(t_f), t_f\right) = 0}$$

NATURAL BOUNDARY CONDITION for Euler equation

Summary:

$$\Delta J(x, \delta x) = \overbrace{\delta J(x, \delta x)}^{\text{linear}} + \overbrace{g(x, \delta x)}^{\to 0}\,\|\delta x\|$$

δJ variation of J $\Rightarrow$ J differentiable

$$\boxed{\delta J(x^*, \delta x) = 0}$$ Fundamental theorem of calculus of variation

THE SIMPLEST VARIATIONAL PROBLEM

$$J(x) = \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t)\,dt$$

Problem 1°. $x(t_0)$, $x(t_f)$, t_0, t_f fixed

$\Rightarrow$

$$(*) \quad \frac{\partial g}{\partial x}(x^*, \dot{x}^*, t) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}}(x^*, \dot{x}^*, t)\right] = 0 \quad , \; x(t_0), x(t_f) \text{ given}$$

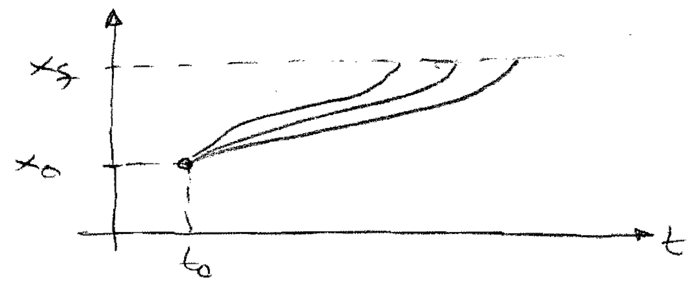
EULER EQUATION

Problem 2° t_0, t_f, $x(t_0)$ fixed; $x(t_f)$ free

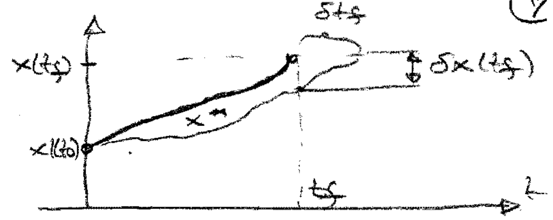
$\Rightarrow$ (*) EE and the natural boundary conditions

$$\frac{\partial g}{\partial \dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f) = 0$$

Problem 3° $x(t_0)$, $x(t_f)$, t_0 fixed, t_f free



$$\Delta J = J(x+\delta x,\ t+\delta t) - J(x,t)$$



$$\left\{\begin{aligned} \Delta J &= \int_{t_0}^{t_f} [g(x^*+\delta x,\ \dot{x}^*+\delta\dot{x},\ t) - g(x^*,\ \dot{x}^*,t)]\,dt \\ &+ \int_{t_f}^{t_f+\delta t_f} g(x(t),\ \dot{x}(t),\ t)\,dt \end{aligned}\right.$$

x^* not defined in $[t_f, t_f+\delta t_f]$

$$\Delta J = \int_{t_0}^{t_f} \Big[\cancel{g(x^*,\dot{x}^*,t)} + \frac{\partial g}{\partial x}(x^*,\dot{x}^*,t)\,\delta x + \frac{\partial g}{\partial \dot{x}}(x^*,\dot{x}^*,t)\,\delta\dot{x} + h.o.t.$$

$$- \cancel{g(x^*,\dot{x}^*,t)}\Big]\,dt + \int_{t_f}^{t_f+\delta t_f} g(x(t),\dot{x}(t),t)\,dt$$

δt_f is small

$$\Delta J = \int_{t_0}^{t_f} \Big[\frac{\partial g}{\partial x}(x^*,\dot{x}^*,t)\,\delta x + \frac{\partial g}{\partial \dot{x}}(x^*,\dot{x}^*,t)\,\delta\dot{x}\Big]\,dt$$

$$+ \ g(x(t_f),\dot{x}(t_f),t_f)\,\delta t_f$$

Integrating by parts

$$\Delta J = \frac{\partial g}{\partial \dot{x}}(x^*(t_f),\dot{x}^*(t_f),t_f)\,\delta x(t_f) - 0 + g(x(t_f),\dot{x}(t_f),t_f)\,\delta t_f$$

$$+ \int_{t_0}^{t_f} \Big\{\frac{\partial g}{\partial x}(x^*,\dot{x}^*,t) - \frac{d}{dt}\Big[\frac{\partial g}{\partial \dot{x}}(x^*,\dot{x}^*,t)\Big]\Big\}\,\delta x(t)\,dt$$

Using

$$g(x(t_f),\dot{x}(t_f),t_f) = g(x^*(t_f),\dot{x}^*(t_f),t_f) + \frac{\partial g}{\partial x}(x^*(t_f),\dot{x}^*(t_f),t_f)\,\delta x(t_f)$$

$$+ \frac{\partial g}{\partial \dot{x}}(x^*(t_f),\dot{x}^*(t_f),t_f)\,\delta\dot{x}(t_f)$$

and taking only linear part (for variation)

$$\delta J = \frac{\partial g}{\partial \dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f)\delta x(t_f) + g(x^*(t_f), \dot{x}^*(t_f), t_f)\,\delta t_f$$
$$+ \underbrace{\int_{t_0}^{t_f} \left\{ \frac{\partial g}{\partial x}(x^*, \dot{x}^*, t) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}}(x^*, \dot{x}^*, t)\right]\right\} \delta x(t)\, dt}_{=0}$$

EULER EQUATION

$$\delta J = \frac{\partial g}{\partial \dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f)\,\delta x(t_f) + g(x^*(t_f), \dot{x}^*(t_f), t_f)\,\delta t_f$$

since

$$\delta x(t_f) < 0 \qquad \text{and} \qquad \dot{x}^*(t_f) \doteq \frac{\delta x(t_f)}{\delta t_f}$$

first approximation

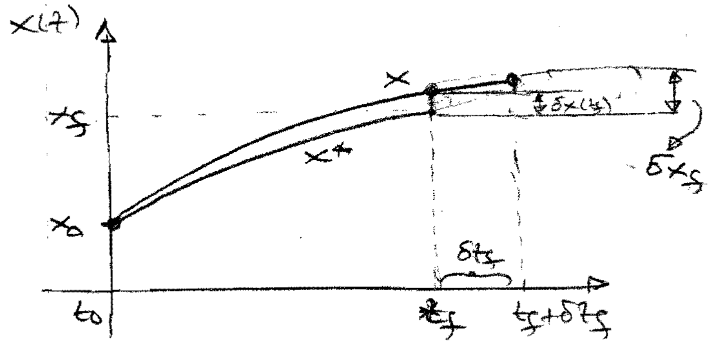
$$\delta J = \left\{ -\frac{\partial g}{\partial \dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f)\,\dot{x}^*(t_f) + g(x^*(t_f), \dot{x}^*(t_f), t_f) \right\} \underbrace{\delta t_f}_{\neq 0}$$

$\Rightarrow$

$$\boxed{-\frac{\partial g}{\partial \dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f)\,\dot{x}^*(t_f) + g(x^*(t_f), \dot{x}^*(t_f), t_f) = 0}$$

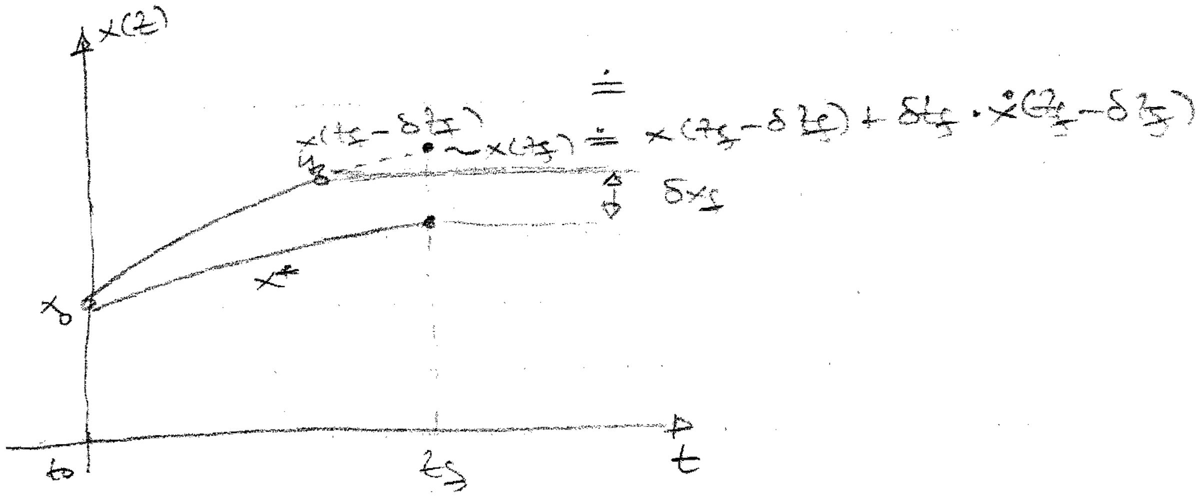
boundary condition at t_f

<u>Problem 4.</u> t_0, $x(t_0)$ fixed t_f and $x(t_f)$ free



$$\delta x_f \neq \delta x(t_f)$$

$$\boxed{\delta x_f = \delta x(t_f) + \dot{x}(t_f)\,\delta t_f}$$



$$\delta x_f = \delta x(t_f) - \delta t_f\,\dot{x}(t_f-\delta t_f)$$

$$\delta x_f \doteq \delta x(t_f) - \delta t_f\,\dot{x}(t_f) + \text{h.o.t}$$

Exactly the same procedure as in Problem 3.

$$\Delta J = \left[\frac{\partial g}{\partial \dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f)\right]\delta x(t_f) + [g(x^*(t_f), \dot{x}^*(t_f), t_f]\delta t_f$$
$$+\int_{t_0}^{t_f}\left\{\frac{\partial g}{\partial x}(x^*(t), \dot{x}^*(t), t) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}}(x^*(t), \dot{x}^*(t), t)\right]\right\}\delta x(t)\,dt + \text{h.o.t}$$

$$\delta x(t_f) = \delta x_f - \dot{x}^*(t_f)\delta t_f$$

$$\delta J(x^*, \delta x) = 0 = \left[\frac{\partial g}{\partial \dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f)\right]\delta x_f +$$
$$+\left[g(x^*(t_f), \dot{x}^*(t_f), t_f) - \frac{\partial g}{\partial \dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f)\dot{x}^*(t_f)\right]\delta t_f$$
$$+\int_{t_0}^{t_f}\underbrace{\left\{\frac{\partial g}{\partial x}(x^*(t), \dot{x}^*(t), t) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}}(x^*(t), \dot{x}^*(t), t)\right]\right\}}_{=0 \quad \text{EE}}\delta x(t)\,dt$$

$\Rightarrow$

$$\boxed{\begin{aligned}0 = &\frac{\partial g}{\partial \dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f)\delta x_f + \\ &+\left[g(x^*(t_f), \dot{x}^*(t_f), t_f) - \frac{\partial g}{\partial \dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f)\dot{x}^*(t_f)\right]\delta t_f\end{aligned}}$$

(end conditions)

① $x(t_f)$ and δt_f unrelated

$$\Rightarrow \frac{\partial g}{\partial \dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f) = 0$$

and
$$g(x^*(t_f), \dot{x}^*(t_f), t_f) - \frac{\partial g}{\partial \dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f)\,\dot{x}^*(t_f) = 0$$

$$\Rightarrow \boxed{g(x^*(t_f), \dot{x}^*(t_f), t_f) = 0}$$

(2) $x(t_f) = \theta(t_f)$ = the final value of x may be constrained to lie on a specified moving point

$\Rightarrow$

$$\boxed{\delta x_f \doteq \frac{d\theta(t_f)}{dt} \cdot \delta t_f}$$ (see Fig. 4-14)

straight line through the origin

$$x(t_f) = \frac{d\theta(t_f)}{dt} \cdot \delta t_f + \text{h.o.t}$$

$\Rightarrow$

$$\boxed{\frac{\partial g}{\partial \dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f)\left[\frac{d\theta(t_f)}{dt} - \dot{x}^*(t)\right] + g(x^*(t_f), \dot{x}^*(t_f), t_f) = 0}$$

TRANSVERSALITY CONDITION

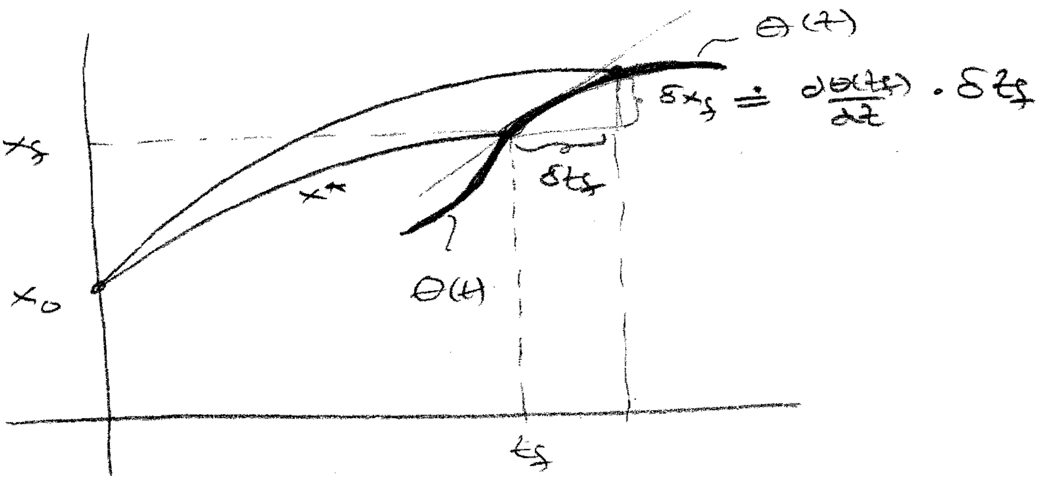


Fig. 4-14