$\boxed{3.10}$ ANALYTICAL RESULTS – DISCRETE LINEAR REGULATOR PROBLEM

$$x(k+1) = A(k)\,x(k) + B(k)\,u(k)$$

$$J = \frac{1}{2} x^T(N) H x(N) + \frac{1}{2} \sum_{k=0}^{N-1} \left[x^T(k) Q(k) x(k) + u^T(k) R(k) u(k) \right]$$

$$Q \geq 0 \qquad\qquad R > 0$$

$$J_{NN}(x(N)) = \underset{\geq 0}{\frac{1}{2} x^T(N) H x(N)} = J^*_{NN}(x(N)) \triangleq \frac{1}{2} x^T(N)\, P(0)\, x(N)$$

$P(0) = H$ notation

$$J_{N-1,N}(x(N-1), u(N-1)) = \frac{1}{2} x^T(N-1)\, Q\, x(N-1)$$

$$+ \frac{1}{2} u^T(N-1) R u(N-1) + \frac{1}{2} x^T(N)\, P(0)\, x(N)$$

$$J^*_{N-1,N}(x(N-1)) \triangleq \min_{u(N-1)} \left\{ J_{N-1,N}(x(N-1), u(N-1)) \right\}$$

$$= \min_{u(N-1)} \Big\{ \frac{1}{2} x^T(N-1) Q x(N-1) + \frac{1}{2} u^T(N-1) R u(N-1) +$$

$$+ \frac{1}{2} \left[A x(N-1) + B u(N-1) \right]^T P(0) \left[A x(N-1) + B u(N-1) \right] \Big\}$$

Assuming that the admissible control is not bounded $\Rightarrow$ STATIC OPTIMIZATION PROBLEM

$$\frac{\partial J_{N-1,N}}{\partial u(N-1)} = 0 = R u(N-1) + B^T P(0) B\, u(N-1)$$

$$+ \frac{1}{2} B^T P(0) A x(N-1) + \frac{1}{2} B^T P(0) A x(N-1)$$

$$\frac{\partial^2 J_{N-1,N}}{\partial^2 u(N-1)} = R + B^T P(0) B > 0 \Rightarrow \text{minimum}$$

the optimal control gives:

$$\boxed{u^*(N-1) = -[R + B^T P(0) B]^{-1} B^T P(0) A\, x(N-1)}$$

$$u^*(N-1) \triangleq F(N-1)\, x(N-1)$$

$$J^*_{N-1,N}(x(N-1)) = \tfrac{1}{2} x^T(N-1) \Big\{ [A + BF(N-1)]^T P(0) [A + BF(N-1)] + F^T(N-1) R F(N-1) + Q \Big\}\, x(N-1)$$

$$\triangleq \frac{1}{2} x^T(N-1) \boxed{P(1)}\, x(N-1)$$

$$u^*(N-k) \triangleq F(N-k)\, x(N-k)$$

$$J^*_{N-k,N} \triangleq \tfrac{1}{2} x^T(N-k) P(k)\, x(N-k)$$

$$P(k) = [A - BF(N-k)]^T P(k-1) [A + BF(N-k)] + F^T(N-k) R F(N-k) + Q$$

$$F(N-k) = (R + B^T P(k-1) B)^{-1} B^T P(k-1) A$$

$k = 0$

$$\Rightarrow \boxed{J^*_{0,N}(x_0) = \tfrac{1}{2} x^T(0) P(N)\, x(0)}$$

$\boxed{3.11}$ THE HAMILTON – JACOBI – BELLMAN EQUATION

$$\dot{x} = a(x(t), u(t), t)$$

$$J = \int_{\boxed{t_0}}^{t_f} g(x(\tau), u(\tau), \tau)d\tau + h(x(t_f), t_f)$$

t_0, t_f fixed

$$J(x(t), t, u(\tau)) = h(x(t_f), t_f) + \int_t^{t_f} g(x(\tau), u(\tau), \tau)\, d\tau$$
$$t \le \tau \le t_f$$

<u>find minimizing control for all $t \le t_f$</u>

$$J^*(x(t), t) = \min_{\substack{u(\tau) \\ t \le \tau \le t+\Delta t}} \left\{ \int_t^{t+\Delta t} g\, d\tau + \int_{t+\Delta t}^{t_f} g\, d\tau + h(x(t_f), t_f) \right\}$$

$$J^*(x(t), t) = \min_{\substack{u(\tau) \\ t \le \tau \le t+\Delta t}} \left\{ \int_t^{t+\Delta t} g\, d\tau + J^*(x(t+\Delta t), t+\Delta t) \right\}$$

$$= \min_{\substack{u(\tau) \\ t \le \tau \le t+\Delta t}} \left\{ \int_t^{t+\Delta t} g\, d\tau + J^*(x, t) + \frac{\partial^* J}{\partial t}(x(t), t)\, \Delta t + \left(\frac{\partial^* J}{\partial x}(x(t), t)\right)^T \underbrace{(x(t+\Delta t) - x(t))}_{\Delta x} + h.o.t. \right\}$$

<u>for Δt small</u>

$$x(t+\Delta t) - x(t) = \Delta t\, a(x(t), u(t), t)$$

$$\cancel{J^*(x(t),t)} = \min_{u(t)} \{ g(x(t),u(t),t)\,\Delta t + \cancel{J^*(x(t),t)}$$
$$+ J_t^*(x(t),t)\,\Delta t + J_x^{*T}(x(t),t)\,a(x(t),u(t),t)\,\Delta t\}$$
$$+ h.o.$$

$$0 = \min_{u(t)} \{ g(x(t),u(t),t)\,\Delta t + J_x^{*T}(x(t),t)\,a(x(t),u(t),t)\,\Delta t\} +$$
$$+ J_t^*(x(t),t)\,\Delta t$$

$$\boxed{0 = J_t^*(x(t),t) + \min_{u(t)} \{ g(x(t),u(t),t) + J_x^{*T}(x(t),t)\,a(x(t),u(t),t)\}}$$

partial differential equation with boundary condition

$$\boxed{J^*(x(t_f),t_f) = h(x(t_f),t_f)}$$

Introducing the Hamiltonian

$$H(x(t),u(t),J_x^*,t) \triangleq g(x(t),u(t),t) + J_x^{*T}(x(t),t)\,a(x(t),u(t),t)$$

and

$$H(x(t),u^*(x(t),J_x^*,t),J_x^*,t) = \min_{u(t)} H(x(t),u(t),J_x^*,t)$$

$\Rightarrow$

$$\boxed{0 = J_t^*(x(t),t) + H(x(t),u^*(x(t),J_x^*,t),J_x^*,t)}$$

Hamilton - Jacobi - Bellman equation

3.12. CONTINUOUS LINEAR REGULATOR PROBLEMS

$$\dot{x} = A(t)\,x + B(t)\,u$$

$$J = \frac{1}{2}x^T(t_f)H\,x(t_f) + \int_{t_0}^{t_f}\frac{1}{2}\left[x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)\right]dt$$

$$H \geq 0, \quad Q \geq 0, \quad R > 0$$

$$\mathcal{H}(x, u, J_x^*, t) = \frac{1}{2}x^TQx + \frac{1}{2}u^TRu + J_x^{*T}(Ax + Bu)$$

<u>necessary conditions:</u>

$$\frac{\partial \mathcal{H}}{\partial u} = 0 = Ru + B^TJ_x^* = 0 \qquad (1)$$

$$\frac{\partial^2 \mathcal{H}}{\partial u^2} = R > 0 \quad \Rightarrow \text{global minimum (sufficient condition)}$$

$$(1) \Rightarrow \boxed{u^* = -R^{-1}B^TJ_x^*}$$

$$\frac{1}{2}x^TQx + \frac{1}{2}J_x^{*T}BR^{-1}B^TJ_x^* + J_x^{*T}Ax - J_x^{*T}BR^{-1}B^TJ_x^*$$

$$= \frac{1}{2}x^TQx - \frac{1}{2}J_x^{*T}BR^{-1}B^TJ_x^* + J_x^{*T}Ax$$

The H-J-B equation is

$$0 = J_t^* + \left(\frac{1}{2}x^TQx - \frac{1}{2}J_x^{*T}BR^{-1}B^TJ_x^* + J_x^{*T}Ax\right)$$

$$J^*(x(t_f), t_f) = \frac{1}{2}x^T(t_f)Hx(t_f)$$

Assuming a solution in the form

$$J^*(x(t),t) = \frac{1}{2} x^T(t)\, K(t)\, x(t) \Rightarrow J_x = K(t)\, x(t)$$

$$\Rightarrow 0 = \frac{1}{2} x^T Q x - \frac{1}{2} x^T K B R^{-1} B^T K x + x^T K A x + \frac{1}{2} x^T \dot{K} x$$

since

$$x^T K A x = \frac{1}{2} x^T K A x + \frac{1}{2} x^T A^T K^T x$$

$\Rightarrow$

$$\frac{1}{2} x^T \underbrace{\left(Q - K B R^{-1} B^T K + K A + A^T K + \dot{K} \right)}_{=0} x = 0$$

$$\boxed{KA + A^T K + Q - K B R^{-1} B^T K = -\dot{K}}$$

since $J^*(x(t_f), t_f) = \frac{1}{2} x^T(t_f)\, K(t_f)\, x(t_f) = \frac{1}{2} x^T(t_f)\, H\, x(t_f)$

$$\boxed{K(t_f) = H}$$

Thus,

$$\boxed{u^*(t) = -R^{-1} B^T J_x^* = -R^{-1} B^T K(t)\, x(t)}$$

HW #2 Due Feb. 18

3.7 ; 3.8 ; 3.10 ; 3.4 ; [3.23 need MATLAB optional]

SUCCESSIVE APPROXIMATIONS FOR SOLVING FUNCTIONAL EQUATION OF DYNAMIC PROGRAMMING

(1) $\frac{dx}{dt} = \dot{x} = f(x, u), \qquad x(0) = x_0$

(2) $J = \int_0^\infty g(x, u)\, dt$

Note this scheme may not converge. Convergence has to be proved for every particular choice of f and g. Good for writing papers. For actual computations use the discrete version of B-J-H eq.

Hamiltonian

$$H(x, p, u) = g(x, u) + p^T f(x, u)$$

Take $u_0(x)$ such that (1) is stabilized

(3) $\dot{x}^{(0)} = f(x^{(0)}, u_0(x^{(0)}))$

and integral (2) exist

(4) $J^{(0)} = \int_0^\infty g(x^{(0)}, u_0(x^{(0)}))\, dt$

Then

$$\dot{p}^{(0)}(t) = -\frac{\partial H}{\partial x} = -\frac{\partial g(x^{(0)}, u(x^{(0)}))}{\partial x} + \frac{\partial f(x^{(0)}, u(x^{(0)}))}{\partial x} \cdot p^{(0)}$$

Use $p^{(0)}(t)$ to initialize the following sequence of optimization problems:

(5) $$\begin{cases} \min_u H^{(0)}(x, u) = \min_u \{g(x, u) + p^{(0)T} f(x, u)\} \Rightarrow u_1(x) \\ \dot{x}^{(1)} = f(x^{(1)}, u^{(1)}(x^{(1)})) \\ \dot{p}^{(1)} = -\frac{\partial g(x^{(1)}, u(x^{(1)}))}{\partial x} + \frac{\partial f(x^{(1)}, u(x^{(1)}))}{\partial x} p^{(1)} \Rightarrow p^{(1)} \end{cases}$$

or in general

$$\begin{cases} \min_u H^{(k)}(x, u) = \min_u (g(x, u) + p^{(k)T} f(x, u)) \Rightarrow u_{k+1}(x) \\ \dot{x}^{(k+1)} = f(x^{(k+1)}, u^{(k+1)}(x^{(k+1)})) \\ \dot{p}^{(k+1)} = -\frac{\partial g}{\partial x}\Big|_{\dots} + \frac{\partial f}{\partial x}\Big|_{\dots} p^{(k+1)} \Rightarrow p^{(k+1)} \end{cases}$$