Chapter 2

PERFORMANCE MEASURE

"A problem well put is a problem half solved."

$$\min_{u \in U} J = \int_{t_0}^{t_f} g(x(t), u(t), t)\,dt + h(x(t_f), t_f)$$

subject to

$$\dot{x} = a(x(t), u(t), t), \quad x(t_0) = x_0, \quad x \in X$$

1) MINIMUM-TIME PROBLEM: to transfer a system from an arbitrary initial state $x(t_0) = x_0$ to a specified target set S in minimum time.

$$J = t_f - t_0 = \int_{t_0}^{t_f} dt \qquad g = 1, \quad h = 0$$

2) TERMINAL CONTROL PROBLEM: to minimize the deviation of the final state of a system from its desired value $r(t_f)$

$$J = \|x(t_f) - r(t_f)\|^2$$

or

$$J = \sum_{i=1}^{n} [x_i(t_f) - r_i(t_f)]^2 = \underbrace{(x(t_f) - r(t_f))^T (x(t_f) - r(t_f))}_{=h}$$

or more general $\qquad g = 0$

$$J = (x(t_f) - r(t_f))^T H (x(t_f) - r(t_f))$$

$H \geq 0$ positive semidefinite matrix

3) MINIMUM-CONTROL EFFORT: to transfer a system from an arbitrary initial state $x(t_0) = x_0$ to a specified target set S, with a minimum expenditure of control ~~effort~~

$$J = \int_{t_0}^{t_f} |u(t)|\, dt$$ — minimum fuel problem

or

$$J = \int_{t_0}^{t_f} \underbrace{u^T(t) R u(t)}_{=g} dt \quad , h = 0$$

4) TRACKING PROBLEM: to maintain the system state $x(t)$ as close as possible to the desired state $r(t)$ in the interval (t_0, t_f)

$$J = \int_{t_0}^{t_f} (x(t) - r(t))^T Q (x(t) - r(t))\, dt$$

more general

$$J = \int_{t_0}^{t_f} [(x(t) - r(t))^T Q (x(t) - r(t)) + u^T(t) R u(t)]\, dt$$

or

$$J = \int_{t_0}^{t_f} \left(\| x(t) - r(t) \|_Q^2 + \| u(t) \|_R^2 \right) dt$$

REGULATOR PROBLEM $r(t) = 0$

Appendix 1 (from Kirk, Prentice Hall, 1970)

1) $(CD)^T = D^T C^T$

2) $z^T M y = (z^T M y)^T = y^T M^T z = \text{scalar}$

3) $y^T P y > 0 \quad \forall y \neq 0 \iff P$ is positive definite

$y^T S y \geq 0 \quad \forall y \iff S$ is positive semidefinite

$\mathrm{Re}\{\lambda(P)\} > 0$

$\mathrm{Re}\{\lambda(S)\} \geq 0$

4) Let $P > 0$ and $S \geq 0 \Rightarrow P + S > 0$

Proof:

$$y^T (P+S) y = \underbrace{y^T P y}_{>0} + \underbrace{y^T S y}_{\geq 0} > 0$$

5) $P > 0 \Rightarrow P^{-1}$ exists. Why $P^{-1} = \frac{1}{\det P} \cdot \mathrm{adj}\, P$

$$\det P = \prod_{i=1}^{n} \lambda_i$$

6) GRADIENT: Let $s(y)$ be a scalar function of $y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$

$$\text{gradient} = \frac{\partial s(y)}{\partial y} = \begin{bmatrix} \frac{\partial s}{\partial y_1} \\ \frac{\partial s}{\partial y_2} \\ \vdots \\ \frac{\partial s}{\partial y_n} \end{bmatrix}$$

7) $\frac{\partial}{\partial y}(y^T M z) = Mz$
$y \in \mathbb{R}^{m\times 1}$, $z \in \mathbb{R}^{m\times 1}$, $M \in \mathbb{R}^{m\times m}$ } HW

8) $\frac{\partial}{\partial y}(y^T M y) = My + M^T y = 2My$ of $M = M^T$

9) Let $a(y)$ be a vector function, $a(y) \in \mathbb{R}^n$, $y \in \mathbb{R}^m$

$$\frac{\partial}{\partial y}(a(y)) = \begin{bmatrix} \frac{\partial a_1}{\partial y_1} & \frac{\partial a_1}{\partial y_2} & \cdots & \frac{\partial a_1}{\partial y_m} \\ \frac{\partial a_2}{\partial y_1} & \frac{\partial a_2}{\partial y_2} & \cdots & \frac{\partial a_2}{\partial y_m} \\ \vdots & & & \\ \frac{\partial a_n}{\partial y_1} & \frac{\partial a_n}{\partial y_2} & \cdots & \frac{\partial a_n}{\partial y_m} \end{bmatrix} = \text{Jacobian}$$

10) Let $s(y)$ be a scalar function of $y \in \mathbb{R}^m$, then

$$\frac{\partial^2 s}{\partial y^2}(y) = \begin{bmatrix} \frac{\partial^2 s}{\partial y_1^2} & \frac{\partial^2 s}{\partial y_1 \partial y_2} & \cdots & \frac{\partial^2 s}{\partial y_1 \partial y_m} \\ \frac{\partial^2 s}{\partial y_2 \partial y_1} & \frac{\partial^2 s}{\partial y_2^2} & \cdots & \frac{\partial^2 s}{\partial y_2 \partial y_m} \\ \vdots & & & \\ \frac{\partial^2 s}{\partial y_m \partial y_1} & \frac{\partial^2 s}{\partial y_m \partial y_2} & \cdots & \frac{\partial^2 s}{\partial y_m \partial y_m} \end{bmatrix}$$

11) $\frac{\partial^2}{\partial u^2}(u^T R u) = 2R$ (HW)