LINEAR-QUADRATIC NASH GAME

$$\dot{x} = Ax + B_1 u_1 + B_2 u_2, \quad x(t_0) = x_0$$

$$J_1 = \frac{1}{2}\int_{t_0}^{t_f} (x^T Q_1 x + u_1^T R_{11} u_1 + u_2^T R_{12} u_2)\,dt + \frac{1}{2} x^T(t_f) F_1\, x(t_f)$$

$$J_2 = \frac{1}{2}\int_{t_0}^{t_f} (x^T Q_2 x + u_1^T R_{21} u_1 + u^2 R_{22} u_2)\,dt + \frac{1}{2} x^T(t_f) F_2\, x(t_f)$$

In the following we will need that $R_{11} > 0$, $R_{22} > 0$.
Even more we assume that

$$Q_1 = Q_1^T \geq 0, \qquad Q_2 = Q_2^T \geq 0, \qquad F_1 = F_1^T \geq 0$$

$$R_{11} = R_{11}^T > 0, \qquad R_{22} = R_{22}^T > 0$$

$$R_{12} = R_{12}^T \geq 0, \qquad R_{21} = R_{21}^T \geq 0$$

The necessary conditions for the Nash optimum imply

$$H_1 = \frac{1}{2}(x^T Q_1 x + u_1^T R_{11} u_1 + u_2^T R_{12} u_2) + p_1^T (Ax + B_1 u_1 + B_2 u_2)$$

$$H_2 = \frac{1}{2}(x^T Q_2 x + u_1^T R_{21} u_1 + u_2^T R_{22} u_2) + p_2^T (Ax + B_1 u_1 + B_2 u_2)$$

(a) $\dot{x} = \dfrac{\partial H}{\partial p_i^T} = Ax + B_1 u_1 + B_2 u_2, \quad x(t_0) = x_0$

(b) $\dot{p}_1 = -\left(\dfrac{\partial H_1}{\partial x}\right)^T - \left(\dfrac{\partial u_2}{\partial x}\right)^T \left(\dfrac{\partial H_1}{\partial u_2}\right)^T, \quad p_1(t_f) = F_1 x(t_f)$

(c) $\dot{p}_2 = -\left(\dfrac{\partial H_2}{\partial x}\right)^T - \left(\dfrac{\partial u_1}{\partial x}\right)^T \left(\dfrac{\partial H_2}{\partial u_1}\right)^T, \quad p_2(t_f) = F_2 x(t_f)$

(d) $0 = \dfrac{\partial H_1}{\partial u_1} = R_{11} u_1 + B_1^T p_1 \Rightarrow u_1^* = -R_{11}^{-1} B_1^T p_1$

(e) $0 = \dfrac{\partial H_2}{\partial u_2} = R_{22} u_2 + B_2^T p_2 \Rightarrow u_2^* = -R_{22}^{-1} B_2^T p_2$

(d) and (e) in (a)-(c) $\Rightarrow$

(a) $\dot{x}^* = Ax^* - \underbrace{B_1 R_{11}^{-1} B_1^T}_{=S_1} p_1^* - \underbrace{B_2 R_{22}^{-1} B_2^T}_{=S_2} p_2^*, \quad x^*(t_0) = x_0$

(b) $\dot{p}_1^* = -Q_1 x^* - A^T p_1^* - \left(\frac{\partial u_2^*}{\partial x}\right)^T \left(B_2^T p_1^* + R_{12} u_2^*\right), p_1(t_f) = F_1 x$

(c) $\dot{p}_2^* = -Q_2 x^* - A^T p_2^* - \left(\frac{\partial u_1^*}{\partial x}\right)^T \left(B_1^T p_2^* + R_{21} u_1^*\right), p_2(t_f) = F_2 x$

Note that if u_1^* and u_2^* do not depend explicitly on x then $\frac{\partial u_2^*}{\partial x} = 0$ and $\frac{\partial u_1^*}{\partial x} = 0$ and the above equations are simplified to

$$\begin{cases} (a)_0 \quad \dot{x} = Ax - S_1 p_1 - S_2 p_2, & x(t_0) = x_0 \\ (b)_0 \quad \dot{p}_1 = -Q_1 x - A^T p_1, & p_1(t_f) = F_1 x(t_f) \\ (c)_0 \quad \dot{p}_2 = -Q_2 x - A^T p_2, & p_2(t_f) = F_2 x(t_f) \end{cases}$$

Such Nash strategies are called the open-loop Nash strategies. In general, we have four possibilities

P1	O	F	O	F
P2	O	O	F	F

where "O" indicates the open-loop strategy and "F" stands for the feedback strategy in which $u_i = u_i(x)$, $i = 1, 2$.
It is obvious that the feedback strategy is more powerful than the open-loop strategy.
In the above equations we have dropped "*" for simplicity.

The <u>open-loop strategy</u> depends only on time, that is, $u_1^* = u_1^*(t)$, $u_2^* = u_2^*(t)$. It is a precomputed strategy for any time instant assuming that the game is all the time on the optimal trajectory $x^*(t)$. However, if the game for any reason is not on $x^*(t)$ then $u_1^*(t)$, at the given t and for any time instant greater than t, will not be optimal any more. The <u>feedback strategy</u> $u_2^* = u_2^*(x^*)$ carries information about the state of the game at any time instant, hence the player who has information about ~~its~~ the game's state at any time can adjust his strategy and make it optimal.

THE OPEN-LOOP NASH STRATEGIES LEAD TO THE SO-CALLED TWO POINT BOUNDARY VALUE PROBLEM.

The initial and terminal conditions are split.

It is very well known in mathematics that such problems (in the linear case) can be solved by using

$$p_1(t) = K_1(t)\,x(t)$$

and

$$p_2(t) = K_2(t)\,x(t)$$

and finding the corresponding equations for $K_1(t)$ and $K_2(t)$. In the case of $(a)_0$-$(c)_0$ we have

$$\dot{p}_1 = \dot{K}_1 x + K_1 \dot{x}$$

$$-Q_1 x - A^T p_1 = \dot{K}_1 x + K_1 (Ax - S_1 p_1 - S_2 p_2)$$

$$-Q_1 x - A^T K_1 x = \dot{K}_1 x + K_1 A x - K_1 S_1 K_1 x - K_1 S_2 K_2 *$$

Since the last equation has to hold for any x, we have

$$\boxed{-\dot{K}_1 = K_1 A + A^T K_1 + Q_1 - K_1 S_1 K_1 - K_1 S_2 K_2}$$

with

$$p_1(t_f) = K_1(t_f)x(t_f) = F_1 x(t_f) \Rightarrow \boxed{K_1(t_f) = F_1}$$

Similarly, from

$$p_2(t) = K_2(t)x(t) \quad \text{and} \quad \dot{p}_2(t) = \dot{K}_2(t)x(t) + K_2(t)\dot{x}(t)$$

we get

$$\boxed{-\dot{K}_2 = K_2 A + A^T K_2 + Q_2 - K_2 S_2 K_2 - K_2 S_1 K_1} \quad \boxed{K_2(t_f) = F_2}$$

These two coupled differential equations, known as coupled differential Riccati equations, can be numerically easily solved by integrating backward in time starting with $K_1(t_f)$ and $K_2(t_f)$.

The corresponding optimal open-loop controls are

$$u_1^*(t) = -R_{11}^{-1} B_1^T \cancel{K_1(t)}\, p_1(t)$$
$$u_2^*(t) = -R_{22}^{-1} B_2^T \cancel{K_2(t)}\, p_2(t)$$

where $p_1(t)$ and $p_2(t)$ are obtained from

$$\begin{bmatrix} \dot{x}^*(t) \\ \dot{p}_1^*(t) \\ \dot{p}_2^*(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A & -S_1 & -S_2 \\ -Q_1 & -A^T & 0 \\ -Q_2 & 0 & -A^T \end{bmatrix}}_{\mathcal{A}} \begin{bmatrix} x^*(t) \\ p_1^*(t) \\ p_2^*(t) \end{bmatrix} \qquad \begin{aligned} x^*(t_0) &= x_0 \\ p_1^*(t_0) &= K_1(t_0)x_0 \\ p_2^*(t_0) &= K_2(t_0)x_0 \end{aligned}$$

$$\begin{bmatrix} x^*(t) \\ p_1^*(t) \\ p_2^*(t) \end{bmatrix} = e^{\mathcal{A}(t-t_0)} \begin{bmatrix} x^*(t_0) \\ p_1^*(t_0) \\ p_2^*(t_0) \end{bmatrix}$$

Assuming that the optimization is performed from $t_0 = 0$ to $t_f = +\infty$, the game can reach the steady state value (assuming that $A - S_1K_1 - S_2K_2$ is an asymptotically stable matrix). In such a case $\dot{K}_1 = 0$, $\dot{K}_2 = 0$ (K_1=const, K_2=const) which leads to the coupled algebraic Riccati equations of the open-loop Nash games

$$\boxed{\begin{aligned} 0 &= K_1A + A^TK_1 + Q_1 - K_1S_1K_1 - K_1S_2K_2 \\ 0 &= K_2A + A^TK_2 + Q_2 - K_2S_2K_2 - K_2S_1K_1 \end{aligned}}$$

These equations, eventhough derived thirty years ago, are still not well understood. The research is still going on about the existence of the stabilizing solutions K_1 and K_2 (note $A - S_1K_1 - S_2K_2$ must be asymptotically stable) and finding efficient algorithms for their numerical solution.

Note that K_1 and K_2 are not symmetric matrices, hence each equation has n^2 unknowns for the total of $2n^2$ unknowns.

Some existence conditions are given in Theorem 6.20 and Corollary 6.7, pages 334-335 of the text book. (Engwerda, 1998).

The optimal trajectory now can be obtained from

$$\dot{x}^* = (A - S_1K_1 - S_2K_2)x^*, \quad x^*(t_0) = x_0$$

$$\Rightarrow \quad x^*(t) = e^{(A - S_1K_1 - S_2K_2)(t - t_0)}\, x^*(t_0)$$