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## 5.5 Dynamic Optimization

### 5.5.2 Continuous-time

goal to minimize an integral performance criterion

$$J(x(t_0), t_0) = \int_{t_0}^{t_f} L(x(t), u(t))\,dt + g(x(t_f))$$

along trajectories of a dynamic system

$$\dot{x}(t) = f(x(t), u(t)), \qquad x(t_0) = x_0$$

$f$ = continuous and continuously differentiable function

$u$ = piecewise continuous function

$x(t)$ = n-dimensional state vector

$u(t)$ = m-dimensional control vector

$g(x(t_f))$ = terminal condition (at $t_f$ certain condition must be satisfied - aircraft on a runway)

$u = u(t)$ = open-loop (or programming) control

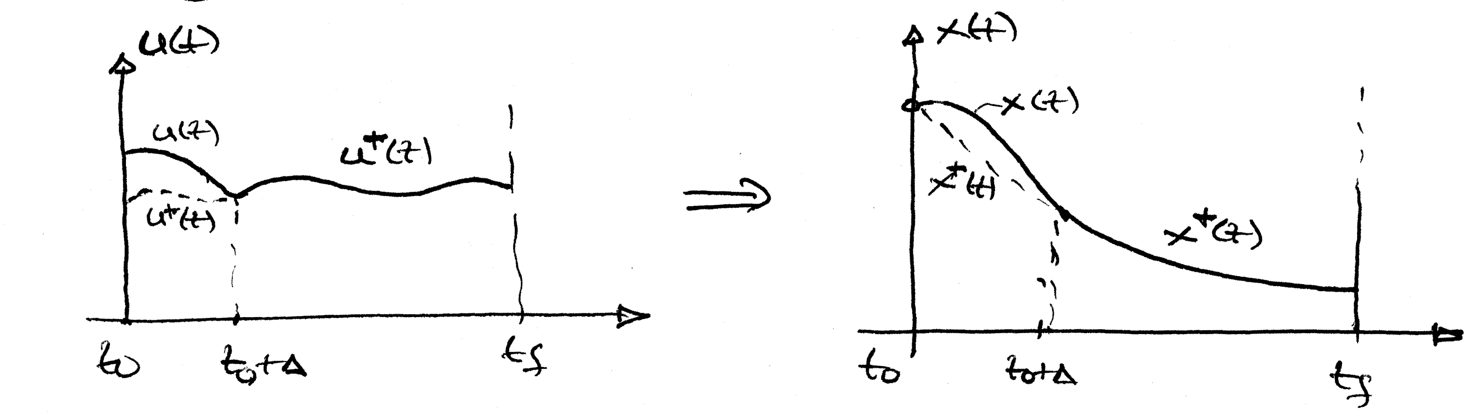
$u = u(x(t))$ = feedback control is desirable since the strategy depends on the state of the system (game)

Assume that an optimal $u^*$ that minimizes $J$ exists. Then

$$J^*(x(t_0), t_0) = \int_{t_0}^{t_f} L(x^*(t), u^*(t))\,dt + g(x^*(t_f))$$

$$\dot{x}^*(t) = f(x^*(t), u^*(t)), \qquad x^*(t_0) = x_0$$

If the optimal control is not applied for every $t \in [t_0, t_f]$ we have




$$J^*(x(t_0), t_0) \le \int_{t_0}^{t_0+\Delta} L(x,u)\,dt + \underbrace{\int_{t_0+\Delta}^{t_f} L(x^*, u^*)\,dt + g(x(t_f))}_{\triangleq\, J^*(x^*(t_0+\Delta),\, t_0+\Delta)}$$

$$J^*(x(t_0), t_0) - J^*(x^*(t_0+\Delta), t_0+\Delta) \le \int_{t_0}^{t_0+\Delta} L(x,u)\,dt$$

$$\lim_{\Delta \to 0} \left\{ - \frac{J^*(x^*(t_0+\Delta), t_0+\Delta) - J^*(x(t_0), t_0)}{\Delta} \right\} \le \lim_{\Delta \to 0} \frac{1}{\Delta} \int_{t_0}^{t_0+\Delta} L(x,u)\,dt$$

$$- \frac{\partial J^*}{\partial t_0}(x^*(t_0), t_0) \le L(x(t_0), u(t_0))$$

since $t_0$ can be taken arbitrary, say $t_0 = t$ we have

$$- \frac{\partial J^*}{\partial t}(x^*(t_0), t_0) \le L(x(t), u(t))$$

If the optimal control is used along the entire trajectory we have the equality, that is

$$\boxed{- \frac{\partial J^*}{\partial t}(x^*(t), t) = L(x^*(t), u^*(t))}$$

Note that

$$-\frac{dJ}{dt}(x(t),t) = -\frac{\partial J}{\partial t} - \frac{\partial J}{\partial x}\cdot\frac{dx}{dt} = -\frac{\partial J}{\partial t} - \frac{\partial J}{\partial x}\cdot f(x,u)$$

Hence, we have

$$\boxed{-\frac{\partial J^*}{\partial t} - \frac{\partial J^*}{\partial x} f(x^*,u^*) = L(x^*,u^*)}$$

This partial differential equation is known as the Hamilton-Jacobi-Bellman equation.

Note that from the original definition of J we have

$$\boxed{J(x(t_f),t_f) = 0 + g(x(t_f))}$$

The above H-J-B equation can be written as

$$-\frac{\partial J^*}{\partial t} = \underbrace{L(x^*,u^*) + \frac{\partial J^*}{\partial x} f(x^*,u^*)}_{\triangleq H\left(x^*,u^*,\frac{\partial J^*}{\partial x}\right) = \text{called Hamiltonian}}$$

$\Rightarrow$

$$-\frac{\partial J^*}{\partial t} = H\left(x^*,u^*,\frac{\partial J^*}{\partial x^*}\right) = \min_u \left\{H\left(x,u,\frac{\partial J}{\partial x}\right)\right\} \qquad (5.25 \text{ text})$$

In the case of time invariant systems with $t_f \to \infty$, and J not explicitly depending on t H-B-J equation has the form

$$0 = H\left(x^*,u^*,\frac{\partial J^*}{\partial x}\right) \Leftrightarrow$$

on the optimal trajectory (under the optimal control) the Hamiltonian is equal to zero

How to solve the H-B-J equation?

$$-\frac{\partial J^*}{\partial t} = \min_u H\left(x, u, \frac{\partial J}{\partial x}\right) = \min_u \left\{L(x^*, u^*) + \frac{\partial J^*}{\partial x} f(x^*, u^*)\right\}$$

$$\frac{\partial H}{\partial u} = 0 \xrightarrow[\text{hopefully}]{} u^* = \varphi\left(x^*, \frac{\partial J^*}{\partial x}\right)$$

going back to H-B-J equation

$$-\frac{\partial J^*}{\partial t} = L\left(x^*, \varphi\left(x^*, \frac{\partial J^*}{\partial x}\right)\right) + \frac{\partial J^*}{\partial x} f\left(x^*, \varphi\left(x^*, \frac{\partial J^*}{\partial x}\right)\right)$$

Partial differential equation (in general very very difficult for solving) whose solutions are

$$\frac{\partial J^*}{\partial t} \quad \text{and} \quad \frac{\partial J^*}{\partial x} \Rightarrow u^* = \varphi\left(x^*, \frac{\partial J^*}{\partial x}\right)$$

Using the definition of the Hamiltonian

$$H\left(x^*, u^*, \frac{\partial J^*}{\partial x}\right) = L(x^*, u^*) + \frac{\partial J^*}{\partial x} f(x^*, u^*)$$

we have

(1) $\dot{x}^* = \frac{\partial H^*}{\partial p}$, $x^*(t_0) = x_0$ $\qquad p = \frac{\partial J}{\partial x}$

(2) $\dot{p}^* = -\frac{\partial H^*}{\partial x}$, $p^*(t_f) = \frac{\partial J^*}{\partial x}(t_f) = \frac{\partial g}{\partial x}(t_f)$

(3) $\frac{\partial H^*}{\partial u} = 0 \Rightarrow u^* = \varphi(x^*, p^*)$

These three equations constitute necessary conditions for the optimum.

Note that in the case of constraints, for example $|u(t)| < \text{const}$, $|x(t)| < \text{const}$, we can not use the unconstrained optimization $\frac{\partial H}{\partial u} = 0$ to find

$$\min_u \{ H(x, u, \frac{\partial J}{\partial x}) \}$$

In that case we have the solution given by the so-called MINIMUM PRINCIPLE OF PONTRYAGIN, which says

$$\boxed{H(x^*, u^*, \frac{\partial J^*}{\partial x}) \leq H(x^*, u, \frac{\partial J^*}{\partial x}) \qquad (3a)}$$

(EXAMPLE) Linear-quadratic optimization

$$\begin{cases} J = \frac{1}{2}\int_0^{t_f} (x^T Q x + u^T R u)\,dt + \frac{1}{2} x^T(t_f) F x(t_f), & R = R^T > 0 \\ \dot{x} = Ax + Bu, \quad x(t_0) = x_0, & Q = Q^T \geq 0,\ F = F^T \geq 0 \end{cases}$$

Hamiltonian:

$$H = \frac{1}{2}(x^T Q x + u^T R u) + p^T(Ax + Bu) \qquad p^T = \frac{\partial J}{\partial x}$$

Necessary conditions:

(1) $\dot{x} = \frac{\partial H}{\partial p} = Ax + Bu$

(2) $\dot{p} = -\frac{\partial H}{\partial x} = -Qx - A^T p$

(3) $\frac{\partial H}{\partial u} = Ru + B^T p = 0 \implies u^* = -R^{-1} B^T p$

Note $\frac{\partial}{\partial x}(x^T M x) = 2Mx$ assuming $M = M^T$

$\frac{\partial}{\partial x}(p^T A x) = A^T p$ A-any compatible matrix

(3) in (1)-(2) $\Longrightarrow$

(1) $\dot{x} = Ax - BR^{-1}B^T p$ , $x(t_0) = x_0$

(2) $\dot{p} = -Qx - A^T p$ , $p(t_f) = Fx(t_f) = \frac{\partial}{\partial x}\left(\frac{\partial g(t_f}{}\right.$

use $BR^{-1}B^T = S$

(1) $\dot{x} = Ax - Sp$ , $x(t_0) = x_0 =$ given

(2) $\dot{p} = -Qx - A^T p$ , $p(t_f) = F\underset{\uparrow \text{unknown}}{x(t_f)}$

This is a two-point boundary value problem (note split terminal and initial conditions)

We solve the problem by looking for its solution in the form $p(t) = P(t)\,x(t)$

$$\Rightarrow \quad \dot{p} = \dot{P}x + P\dot{x}$$

$$-Qx - A^T p = \dot{P}x + PAx - PSPx$$

$$-\dot{P}x = Qx + A^T Px + PAx - PSPx$$

since this has to hold for every $x(t)$ we get

$$\boxed{-\dot{P} = A^T P + PA + Q - PSP}$$

$$p(t_f) = P(t_f)\,x(t_f) = F\,x(t_f) \Rightarrow \boxed{P(t_f) = F}$$

This is the famous differential Riccati equation. At steady state ($t_f \to \infty$ and constant matrices) we have

$$\boxed{0 = A^T P + PA + Q - PSP}$$ Algebraic Riccati equations.

There are close to one thousand journal papers written about the Riccati equation (most of them on algebraic one)