4.3 STATIC NASH GAMES IN EUCLIDEAN SPACES

We have addressed this issue some time ago (for N=2)
In general case with N-players, the Nash equilibrium is defined by

$$J_1(u_1^*, u_2^*, \ldots, u_N^*) \le J_1(u_1, u_2^*, u_3^*, \ldots, u_N^*)$$
$$J_2(u_1^*, u_2^*, \ldots, u_N^*) \le J_2(u_1^*, u_2, u_3^*, \ldots, u_N^*)$$
$$\vdots$$
$$J_N(u_1^*, u_2^*, \ldots, u_N^*) \le J_N(u_1^*, u_2^*, \ldots, u_{N-1}^*, u_N)$$

$$(u_1^*, u_2^*, \ldots, u_n^*) = u^* = u^N = \text{Nash equilibrium}$$

The rational behavior of each player is to minimize his own losses. In that respect, the only rational behavior for each player is to "partially" minimize his own J, that is

$$\left.\begin{aligned}
\frac{\partial J_1(u_1, u_2, \ldots, u_N)}{\partial u_1} &= 0 = \varphi_1(u_1, u_2, \ldots, u_N) = 0 \\
\frac{\partial J_2(u_1, u_2, \ldots, u_N)}{\partial u_2} &= 0 = \varphi_2(u_1, u_2, \ldots, u_N) = 0 \\
&\vdots \\
\frac{\partial J_N(u_1, u_2, \ldots, u_N)}{\partial u_N} &= 0 = \varphi_N(u_1, u_2, \ldots, u_N) = 0
\end{aligned}\right\} \quad (I)$$

We are faced with a very tough problem of finding solutions to a system of nonlinear algebraic equations. In general there are only three methods to solve algebraic equations

a) Newton method (hardly converges since it heavily depends on the initial conditions)

b) Fixed point iterations

c) Homotopy (too much complicated - it converts the system of algebraic equations into the system of differential equations.

Hence, the fixed point iterations are the only alternative (general one) that we have to solve system (I).

Another problem is that the algebraic equations produce huge number of solutions. For example, let f_i be quadratic polynomial equations then the total number of solutions is $2^{2^{(N-1)}}$ (of course some of the solutions are complex). This is in addition a numerical problem for actual findings of solutions.

However, we just need to analytically show that at least one solution to (I) exist. In that respect we need to formulate (state) one of the fixed point theorems.

BROUWER'S FIXED POINT THEOREM

Let $x \in S$ and $x = f(x)$, where S is a compact (closed and bounded) and convex set, and $f(x)$ is a continuous mapping. Then

$$x^{(i+1)} = f(x^{(i)}) \Longrightarrow x^{(\infty)} = f(x^{(\infty)})$$

that is, the fixed point iterations converge and there exists at least one $x^{(\infty)} \in S$.

Note that the fixed point theorem provides in general the existence of many solutions, but this theorem (or similar fixed point theorems) is the only tool that we have to cope with the general problem of solving algebraic equations.

We have already observed that having several candidates for the Nash equilibrium solution causes problems: which one to choose? how to compare N-tuples.

<u>DEFINITION 4.3</u> Rational Reaction Sets (curves)

Let $u_i \in U_i$ denote strategies and corresponding strategy spaces. Let the minimum of J_i with respect to u_i be attained for $u_{-i} \triangleq \{u_1, u_2, -, u_{i-1}, u_{i+1}, --, u_N\}$ then the set

$$R_i(u_{-i}) = \{\xi_i \in U_i : J_i(\xi_i, u_{-i}) \le J_i(u_i, u_{-i}), \ \forall u_i \in U_i\}$$

is called <u>the rational reaction set of player i</u>.

If $R_i(u_{-i})$ is a singleton, then we have the reaction curve. Note that on the reaction curve

$$u_i = R_i(u_{-i}) \in U_i$$

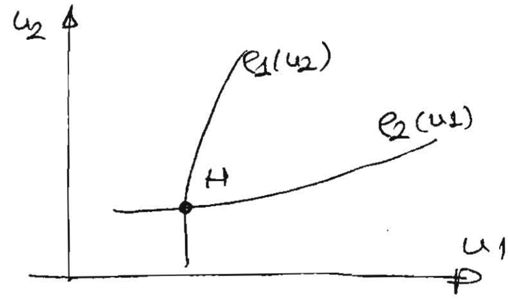
In order to distinguish the reaction sets and reaction curves we will use $\ell_i(u_{-i})$ for <u>the reaction curve</u>. Hence

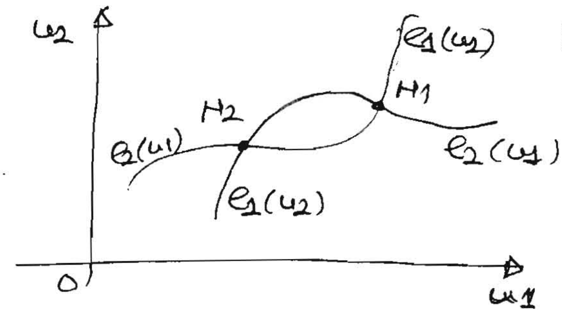
$$u_i = \ell_i(u_{-i}) = \ell_i(u_1, u_2, --, u_{i-1}, u_{i+1}, --, u_N)$$

Note that with DEFINITION 4.3 we have moved far ahead of the original problem defined in (I).

Namely, this definition assumes that (I) has UNIQUE ANALYTICAL SOLUTIONS for every $\varphi_i(u_1, u_2, -, u_N) = 0$
(much stronger assumption than the existence of the unique (numerical) solution)
This is a very strong assumption.

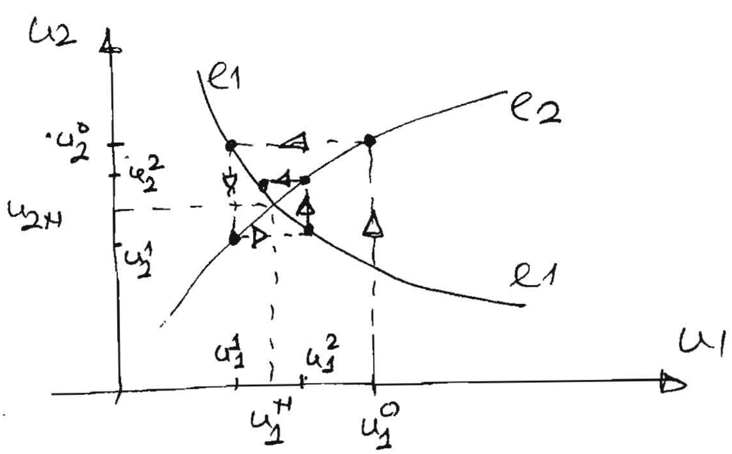
Having reaction curves obtained (assumed) we know that the Nash equilibrium points are obtained at the points of intersections of the reaction curves (recall lecture 2)





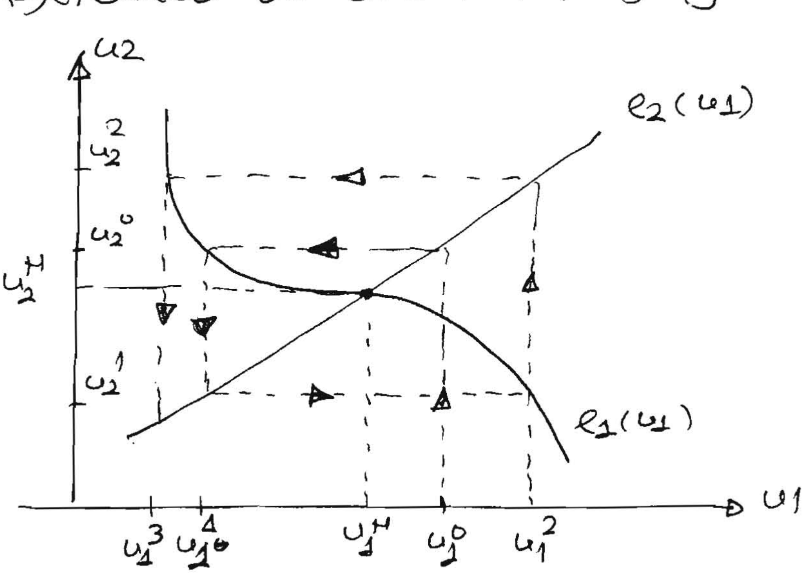
Here, we will not repeat what we did before.

Stable Nash Equilibrium Point



If one of the players (say u_1) deviates from its u_1^H ~~then~~ and plays u_1^0 then P2 will optimize ~~its self~~ his performance and play $u_2^0 = e_2(u_1^0)$. Now P1 plays in an optimal way $u_1^1 = e_1(u_2^0)$ and so on

If the equilibrium is stable this optimal adjustment procedure leads back to the equilibrium. However, if the Nash equilibrium is unstable, the mutual deviation will lead to a permanent divergence, as demonstrated in the next figure



$\boxed{\text{Theorem 4.3}}$ Let $u_i \in U_i$, $i = 1, 2, \dots, N$, U_i compact (closed and bounded) and convex sets of a Euclidean space. Let J_i be continuous in all its arguments and strictly convex ~~for~~ in u_i for every u_j. Then, the corresponding Nash game admits a Nash equilibrium solution.

Proof of this theorem goes along the following lines:

- STRICT CONVEXITY $\Rightarrow$ there exists a unique mapping l_i such that $u_i = l_i(u_1, u_2, \dots, u_{i-1}, u_{i+1}, \dots, u_N)$ uniquely minimizes $J_i(u_1, u_2, \dots, u_N)$

· Hence, we have

$$u = L(u), \quad u = (u_1, u_2, \dots, u_N) \quad L = (l_1, l_2, \dots, l_N)$$

- Need to establish continuity of $L(u)$, which in addition that $u_i \in U_i$ = compact and ~~convex~~ convex set implies the existence of a fixed point (by the Brouwer ~~fixed~~ fixed point theorem) for the algebraic equation $u = L(u)$, that is

$$u^{(i+1)} = L(u^{(i)}), \; i = 1, 2, \dots \Rightarrow u^{(\infty)} = L(u^{(\infty)})$$

$u^{(\infty)}$ = fixed point.

$\boxed{\text{COROLLARY 4.2}}$ Let $u_i \in E^n$ = finite dimensional Euclidean space and J_i be continuous in all its arguments and strictly convex in u_i for every $u_j \in E^n$. Furthermore, let

$$J_i(u_1, u_2, \dots, u_{i-1}, u_i, u_{i+1}, \dots, u_N) \to \infty \quad \text{as } |u_i| \to \infty$$

then a Nash equilibrium exists.

Note by extending strict convexity of J_i to the whole E^n with $u_i \in E^n$ the condition that U_i's are compact and convex is eliminated.

UNIQUENESS OF NASH EQUILIBRIA

$u = L(u) \Rightarrow$ the unique solution $u = u^N$

If the following holds

$$\| L(u^{(i)}) - L(u^{(i-1)}) \| < \rho \, \| u^{(i)} - u^{(i-1)} \| \quad , 0 \leq \rho < 1$$

we have the contraction mapping property and the fixed point iterations $u^{(i+1)} = L(u^{(i)})$ converge to the unique solution.

Similar arguments are established in Proposition 4.1 for a two-person Nash game (to admit a unique solution)

We can also introduce mixed Nash strategies in ~~fin~~ infinite dimensional ~~the~~ static Nash games, ~~but~~ however, our major problem is not to extend the strategy space, but to locate a meaningful Nash solution (hopefully the unique one).

<u>Definition 4.6</u> Assuming that the reaction set of the follower is singleton (hence, we have the unique reaction curve), the strategy $u_1^* \in U_1$ is the Stackelberg strategy for the leader if

$$J_1^* = J_1^*(u_1^*, \ell_2(u_1^*)) \leq J_1(u_1, \ell_2(u_1)), \quad \forall u_1 \in U_1$$

The reaction curve of the follower is defined by

$$\ell_2(u_1) = \{\xi_2 \in U_2 ;\ J_2(u_1, \xi_2) \leq J_2(u_1, u_2), \quad \forall u_2 \in U_2\}$$

In the case of a multivalued reaction set $R_2(u_1)$ the above definition has to be modified into

$$J_1^* = \sup_{u_2 \in R_2(u_1)} J_1(u_1^*, u_2) \leq \sup_{u_2 \in R_2(u_1)} J_1(u_1, u_2)$$

sup (minimal upper value) has to be used instead of min ~~since~~ since now P1 has to find his secured level of losses. Since u_2 has several expressions, say $u_2 = \ell_2^1(u_1)$, $u_2 = \ell_2^2(u_1)$, ... $u_k = \ell_2^k(u_1)$ for each of them the leader has to solve the game (to find ~~the~~ $\underline{\max_{u_1^k} J_1(u_1, \ell_2^j(u_1))}$, $j = 1, 2, \dots, k$) and to choose the strategy that produces the minimum upper (max) value. (see the bimatrix example on page 3, when the follower's response is not unique).

Definition 4.7 ε-Stackelberg Strategy

We have seen in finite Stackelberg games that the Stackelberg Strategy always exists.

In infinite Stackelberg games, however, the solution to the leaders optimization problem

$$\min_{u_1} \left\{ J_1(u_1, u_2) + \lambda \frac{\partial J_2(u_1, u_2)}{\partial u_2} \right\}$$

Lagrange multipliers

does not necessarily exist in all cases.

We can expand the definition of the Stackelberg equilibrium by introducing the concept of ε-Stackelberg equilibrium as follows

$$\sup_{u_2 \in R_2(u_{1\varepsilon}^*)} J_1(u_{1\varepsilon}^*, u_2) \leq J_1^* + \varepsilon \quad , \quad \varepsilon > 0$$

small positive number

MIXED STACKELBERG STRATEGIES are useful since we know that in finite static games

(in mixed strategies) $\bar{J}_1^* \leq J_1^*$ (in pure strategies)

"$\_$" denotes the average cost.

The same holds in infinite static Stackelberg games (See Proposition 4.2).

However, we need to know measure theory in order to define and study mixed Stackelberg strategies in Euclidean spaces, also stochastic calculus (Ito integral, Stratonovich integral, and so on

Example 4.4 shows that the Stackelberg game with strictly convex cost does not have a solution in pure-strategies, but it does in mixed strategies (no need to go over this example, only its conclusion)

4.6 QUADRATIC STACKELBERG GAMES: (AND NASH)

When J_i's are quadratic functions of strategies u_i's. (Section 4.6 - with applications to microeconomics (oligopoly markets)).

This can be an interesting term paper (some knowledge of linear algebra is required).

Another term paper can be written on Braess Networking Paradox (with some further analysis, ~~examples~~), which we presented in ~~time~~ class last time (Section 4.7)

4.2 ε EQUILIBRIUM SOLUTIONS

We have seen in chapters 2 and 3 that for finite games in normal and extensive form a noncooperative equilibrium solution always exists in mixed strategies.

For infinite games that is not always the case and we have to weakened the concept of equilibrium and introduce the so-called ε-equilibrium. (example 4.1).

Definition 4.1 ε-NASH EQUILIBRIUM

Let P1, P2, --, PN have the cost functions $J_1, J_2, --, J_N$
For $\varepsilon \geq 0$ and N-tuple of strategies $\{u_{1\varepsilon}^*, u_{2\varepsilon}^*, --, u_{N\varepsilon}^*\}$

$$\boxed{J_i(u_{1\varepsilon}^*, --, u_{N\varepsilon}^*) \leq \inf J_i(u_{1\varepsilon}^*, u_{2\varepsilon}^*, --, u_{(i-1)\varepsilon}^*, u_i, u_{(i+1)\varepsilon}^*, \ldots) + \varepsilon}$$

for every $i = 1, 2, --, N$. inf = max (lower bound)
is called ε-Nash equilibrium

Definition 4.2 ε-saddle point equilibrium

$$J(u_{1\varepsilon}^*, u_{2\varepsilon}) \leq J(u_{1\varepsilon}^*, u_{2\varepsilon}^*) \leq J(u_{1\varepsilon}, u_{2\varepsilon}^*) + \varepsilon$$

Note that the zero-sum games with simultaneous decision making are a special case of nonzero sum Nash games ($J_1 = -J_2$). Hence Theorem 4.3 holds for zero-sum games as well. Also, Stackelberg for $J_1 = -J_2 \Rightarrow$ zero-sum games with sequential decision making.

Theorem 4.5 Existence of a unique saddle-point equilibrium in zero-sum games

$J(u_1, u_2)$ continuous in u_1 and u_2, strictly convex in u_1 and strictly concave in u_2. U_1 and U_2 convex and compact
$\Rightarrow$ the unique saddle point exists. If $U_1 = U_2 = E^n$, need
$J(u_1, u_2) \to \infty$ as $u_1 \to \infty$ and $J(u_1, u_2) \to -\infty$ as $u_2 \to -\infty$