3.6 STACKELBERG EQUILIBRIUM SOLUTION

One of the players dominate the decision process and enforces his strategy on the other player. That player is called the <u>leader</u> and the other player is called the <u>follower</u>. This is basically <u>hierarchical decision making</u>.

Bimatrix Game Example:

$$A = \begin{array}{c} L \\ M \\ R \end{array} \begin{bmatrix} \boxed{0}^{S_1} & 2 & 3/2 \\ 1 & 1 & 3 \\ -1 & 2 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} \boxed{-1}^{S_1} & 1 & -2/3 \\ 2 & 0 & 1 \\ 0 & 1 & -1/2 \end{bmatrix} \begin{array}{c} L \\ M \\ R \end{array}$$

$$\begin{array}{ccc} L & M & R \end{array} \qquad\qquad \begin{array}{ccc} L & M & R \end{array}$$

Let P1 be the leader. He has information about both matrices A (his losses) and B (the other player losses.

P1 solves the game as follows:

If $i_1 = 1$, then P2 plays $j=1$ since it gives the best outcome for him of -1.

If P1 plays $i=2$, then P1 concludes that P2 plays $j=2$ giving to P2 the value of 0 and to P1 the value of 1.

For $i=3$, P2 will play $j=3$ producing the game value of $(2, -1/2)$.

Since P1 is the leader, he observes that the best strategy for him is $i=1$ (comparing the circled values).. Hence, the Stackelberg strategies are $i=1$, $j=1$ and the value of the game is $(0,-1)^{S_1}$

If P2 is the leader we have the following situation

$$A=\begin{bmatrix} 0 & 2 & \boxed{3/2}^{S_2} \\ 1 & 1 & 3 \\ -1 & 2 & 2 \end{bmatrix} \qquad B=\begin{bmatrix} -1 & 1 & \boxed{-2/3}^{S_2} \\ 2 & 0 & 1 \\ 0 & 1 & -1/2 \end{bmatrix}$$

P2 concludes: if $j=1$ then $i=3$ is the best for P1 hence $(-1,0)$ is the game value

If $j=2$, then $i=2$ is the best for P1 $\Rightarrow$ $(1,0)$

If $j=3$, then $i=1$ " P1 $\Rightarrow$ $(3/2, -2/3)$

Now P2 as the leader chooses among his game values $0, 0, -2/3$, the value of $-2/3$ that is $j=3 \Rightarrow i=3 \Rightarrow (3/2, -2/3)^{S_2}$

Note that the unique Nash equilibrium here is $(1,0)^{N}_{i=2,\, j=2}$, hence in summary

$(i=1, j=1) \Rightarrow (0,-1)^{S_1}$

$(i=3, j=3) \Rightarrow (3/2, -2/3)^{S_2}$

$(i=2, j=2) \Rightarrow (1,0)^{N}$

It can be seen that every leader does better with the Stackelberg than with the Nash strategies.

In the previous example the follower's response to the leaders strategy was unique. Consider now an example when the <u>follower has several alternatives</u> to the leader's strategy so that the leader can not uniquely determine his cost, but he can find his <u>secured cost level</u>. In such a case (non unique follower's response) there is no guarantee that the Stackelberg strategy will be better than the Nash strategy.

$$A = \begin{array}{cc} & \begin{array}{ccc} j=1 & j=2 & \end{array} \\ \begin{array}{c} i=1 \\ \\ \end{array} & \begin{bmatrix} 0 & 1 & 3 \\ 2 & 2 & 1 \end{bmatrix} \end{array}, \qquad B = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & -1 \end{bmatrix}$$

<u>P1 leader</u>. If $i=1$ then $j=1$ or 2 and the secured level for P1 is $\max(0,1)=1$.
If $i=2$ then $j=1$ or 3 leading to the secured level of $\max(2,1)=2$. Hence, P1 will choose $i=1$ since that strategy guarantees that he can not loose more than one. Whatever P2 chooses (under assumption $i=1$) his cost will be 0 ($j=1$ or 2)
Hence, the secured equilibrium $(1,0)$ is obtained for ($i=1$, $j=1$ or 2).
Note that P1 has first to <u>find the max element subject to rational strategies of P2</u> and then to perform minimization and choose the row in which such a max element is minimal.

Now we are ready to formally define <u>the Stackelberg strategies for a two-person finite games</u>

$\boxed{\text{Definition 3.26}}$ (REACTION SETS)
Let $u_1 \in U_1$ and $u_2 \in U_2$ be strategies of P1 and P2. The set $R_2(u_1) \in U_2$ defined below represents the rational reaction set of P2 to the strategy $u_1 \in U_1$ of P1

$$R_2(u_1) = \left\{ u_2 = r_2(u_1) \in U_2 : J_2(u_1, r_2(u_1)) \leq J_2(u_1, u_2),\ \forall u_2 \in U_2 \right\}$$

Note that for any u_1, $r_2(u_1) = u_2$ is not necessarily unique as demonstrated in the previous example.

In general case, when $R_2(u_1)$ for a given u_1 has more than one element the Stackelberg strategy provides the secured equilibrium cost for P1. It is formally defined as follows.

$\boxed{\text{Definition 3.27}}$ SECURED EQUILIBRIUM STACKELBERG STRAT

$$J_1^* = \min_{u_1 \in U_1} \left\{ \max_{u_2 \in R_2(u_1) \subset U_2} J_1(u_1, u_2) \right\} = \begin{cases} \text{optimal secured} \\ \text{equilibrium for} \\ \text{the leader} \end{cases}$$

Such a strategy u_1 is ~~the~~ Stackelgerg strategy for the leader.

Note that by construction ~~the~~ a Stackelberg strategy for the leader always exists, but it is not necessarily unique.

If the follower has the unique response ($R_2(u_1)$ contains only one element for any u_1), then the max part in Definition 3.28 is redundant and the actual game cost for the leader is defined by

$\boxed{\text{Def. 3.27a}}$

$$J_1^* = \min_{u_1 \in U_1} \{ J_1(u_1, r_2(u_1)) \} \implies u_1^*$$

$\boxed{\text{Def. 3.28}}$ Let $u_1^* \in U_1$ then $u_2^* = r_2(u_1) \in U_2$

The pair (u_1^*, u_2^*) constitutes ~~the~~ Stackelberg strategy with the costs $J_1(u_1^*, u_2^*)$ and $J_2(u_1^*, u_2^*)$ representing the Stackelberg game outcomes.

$\boxed{\text{Proposition 3.16}}$ If $R_2(u_1)$ is a singleton then

$$J_1^* \leq J_1^N$$

We can conclude that in Stackelberg strategies the leader solve the game for both players, and then chooses the best solution for himself ~~and~~ imposes on P2 (follower) the binding strategy $u_2^* = T_2(u_1^*)$, which is also optimal for P2 subject to the choice of u_1^*.

- We find the Stackelberg solution by searching rows and columns of matrices A and B as demonstrated on two examples. The Stackelberg solution of a finite game in the extensive form can be found by converting that game into its normal form.

- We will not study multi-act Stackelberg games in extensive form and the corresponding feedback Stackelberg strategies

MIXED STACKELBERG STRATEGIES

In the case of Stackelberg strategies for ~~ze~~ finite games, an equilibrium solution always exists. However, using the mixed strategies the leader can do better than with pure strategies. ("confusing the follower").

(Example)

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1/2 & 1 \\ 1 & 1/2 \end{bmatrix}$$

P1 leader $\Rightarrow$ $(1, 1/2)$ is the Stackelberg equilibrium obtained for $(i=1, j=1)$ and $(i=2, j=2)$

If P1 uses mixed strategies with probabilities $(y_1, y_2) = (1/2, 1/2)$ then the rational choice for P2 is to use mixed strategies with probabilities $(z_1, z_2) = (1/2, 1/2)$, which implies the average costs

$$\bar{J}_1^*(1/2, 1/2) = 1/2 \quad \text{and} \quad \bar{J}_2^*(1/2, 1/2) = 3/4$$

Note that in pure strategies $J_1^* = 1$, $J_2^* = 1/2$. Hence P1 gained and P2 (follower) lost using the mixed strategies.

Formal definitions of mixed strategies are as follows: Let $Y = \{y_i \geq 0, \sum_{i=1}^{m} y_i = 1\}$ and $Z = \{z_i : z_i \geq 0, \sum_{i=1}^{n} z_i = 1\}$.

$\boxed{\text{Definition 3.31}}$ REACTION SET IN MIXED STRATEGIES

$$\bar{R}_2(y) = \{z^0 \in Z : y^T B z^0 \leq y^T B z, \ \forall z \in Z\}$$

is the rational reaction set of P2 in mixed strategies to the mixed strategy $y \in Y$ of P1.

$\boxed{\text{Definition 3.32}}$ STACKELBERG EQUILIBRIUM IN MIXED STRATEGIES

In a bimatrix game a mixed strategy $y^* \in Y$ is a Stackelberg equilibrium strategy if

$$\bar{J}^* = \inf_{y \in Y} \left\{ \max_{z \in \bar{R}_2(y)} \{y^T A z\} \right\}$$

$\bar{J}^*$ is the Stackelberg average cost for the leader in mixed strategies.

$\boxed{\text{Proposition 3.18}}$ I

$$\bar{J}_1^* \leq J_1^*$$

Example

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1/2 & 1 \\ 1 & 1/3 \end{bmatrix}$$

Note that $(1, 1/2)$ and $(1, 1/3)$ are Stackelberg equilibria in pure strategies, obtained respectively for $(i=1, j=1)$ and $(i=2, j=2)$.

Let us find the rational reaction set in mixed strategies for P2

$$\bar{J}_2 = y^T B z = (y_1 \;\; 1-y_1) \begin{pmatrix} \frac{1}{2} & 1 \\ 1 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} z_1 \\ 1-z_1 \end{pmatrix}$$

$$= \frac{1}{3} + \frac{2}{3} z_1 + \frac{2}{3} y_1 - \frac{7}{6} y_1 z_1$$

$$\boxed{\bar{J}_2(y_1, z_1) = \frac{1}{3} + \frac{2}{3} y_1 + \left(\frac{2}{3} - \frac{7}{6} y_1\right) z_1}$$

$$\min_{z_1} \bar{J}_2(y_1) = \min_{z_1} \left\{ \frac{1}{3} + \frac{2}{3} y_1 + \underbrace{\left(\frac{2}{3} - \frac{7}{6} y_1\right)}_{\geq 0 \text{ for } y_1 < \frac{4}{7}} z_1 \right\}$$

$$\bar{R}_2(y_1) = \begin{cases} z = (1,0) & \text{for } y_1 > \frac{4}{7} \\ z = (0,1) & \text{for } y_1 < 4/7 \\ \cancel{z_0}\; z & \text{for } y_1 = 4/7 \end{cases}$$

For P1 we have

$$\bar{J}_1(y, z) = (y_1 \;\; 1-y_1) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ 1-z_1 \end{pmatrix} = y_1 z_1 + (1-y_1)(1-z_1)$$

$$\bar{J}_1(y_1, z_1) = 1 - z_1 - y_1 + 2 y_1 z_1$$

Know that for $\underline{y_1 > \frac{4}{7}}$ $\quad z_1 = 1 \Rightarrow \underline{J_1(y_1, 1) = y_1}$

Also, for $\underline{y_1 < \frac{4}{7}}$ we have $z_1 = 0 \Rightarrow \underline{J_1(y_1, 0) = 1 - y_1}$

Take $y_1 = \frac{4}{7} - \varepsilon$, $\varepsilon > 0$ $\Rightarrow \bar{J}_1\left(\frac{4}{7} - \varepsilon, 0\right) = 1 - \frac{4}{7} + \varepsilon = \frac{3}{7} + \varepsilon$

Hence, the leader prefer to choose

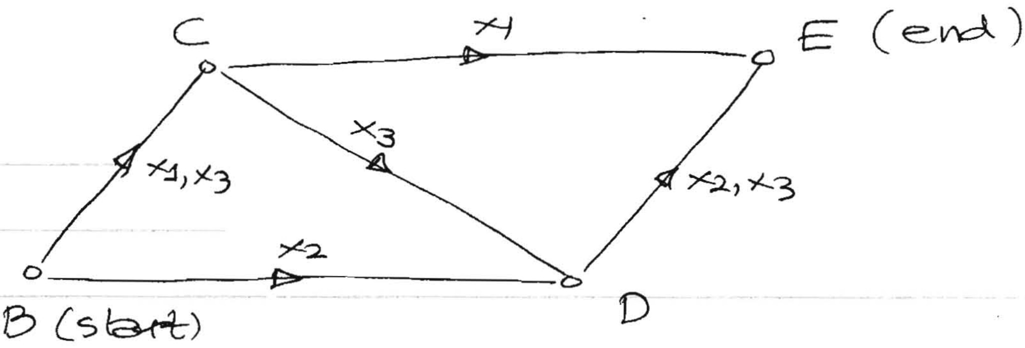
$$(y_1, y_2) = \left(\frac{4}{7} - \varepsilon, \frac{3}{7} + \varepsilon\right) \Longrightarrow \bar{J}_1\left(\begin{pmatrix} \frac{4}{7} - \varepsilon \\ \frac{3}{7} + \varepsilon \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \frac{3}{7} + \varepsilon < J_1^* = 1$$

ε = arbitrary small positive number

Part II
MODERN GAME THEORY
Chapter 4

BRAESS PARADOX (Section 4.7) NETWORKING Exam

In which the Nash equilibrium solution leads to a surprising phenomenon.



Roads with one-way trafic.
All cars (messages, packets) travel along one of the routes:

Route 1: BCE

Route 2: BDE

Route 3: BCDE

The time needed to travel along a segment is given by

BC: $t = 10i$

BD: $t = 50 + i$

CE: $t = 50 + i$

CD: $t = 10 + i$

DE: $t = 10i$

where i = intensity indicate the number of cars per time unit that chose that segment

Let $x_1 + x_2 + x_3 = \text{const}$

where x_j = number of cars that choose route j

The total time needed to go from B to E is a performance criterion for each route

(1) $J_1(x_1,x_3) = 10(x_1+x_3) + 50 + x_1 = 11x_1 + 10x_3 + 50$

(2) $J_2(x_2,x_3) = 50 + x_2 + 10(x_2+x_3) = 11x_2 + 10x_3 + 50$

(3) $J_3(x_1,x_2,x_3) = 10(x_1+x_3) + 10 + x_3 + 10(x_2+x_3) = 10x_1 + 10x_2 + 21x_3 + 10$

The equilibrium is achieved if $J_1 = J_2 = J_3$

$(1) = (2) \Rightarrow \quad 11x_1 = 11x_2 \Rightarrow \quad \boxed{x_1 = x_2} \quad (I)$

$(1) = (3) \Rightarrow \quad \boxed{x_1 - 10x_2 - 11x_3 = -40} \quad (II)$

$(2) = (3) \Rightarrow \quad \boxed{-10x_1 + x_2 - 11x_3 = -40} \quad (III)$

(I) in (II) and (III) imply

$$-9x_1 - 11x_3 = -40$$
$$-9x_1 - 11x_3 = -40$$

Hence we have

$$\left.\begin{array}{l} x_1 = x_2 \\ 9x_1 + 11x_3 = 40 \end{array}\right\}$$

in addition
$x_1 + x_2 + x_3 = \text{const}$
$2x_1 + x_3 = \text{const}$

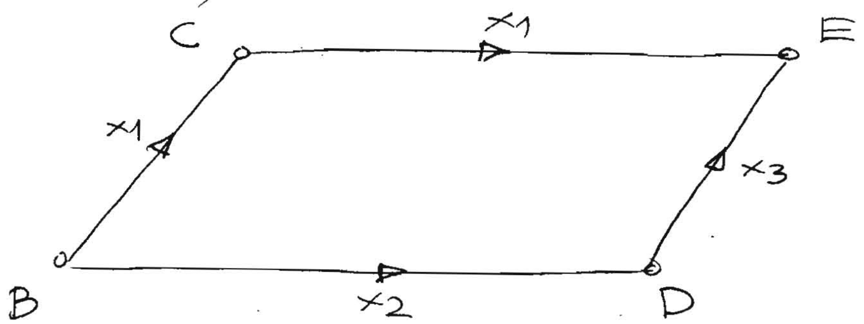
take const $= 6$ (to get nice numbers (integers) for the solution

$$\Rightarrow \left.\begin{array}{l} 9x_1 + 11x_3 = 40 \\ 2x_1 + x_3 = 6 \end{array}\right\} \Rightarrow \boxed{x_1^* = 2} \boxed{x_2^* = 2} \boxed{x_3^* = 2}$$

$J_1^* = J_2^* = J_3^* = 11 \cdot 2 + 10 \cdot 2 + 50 = 92$ time units

Note that ~~this~~ is equilibrium is stable

If we close the path CD (reduces the network capacity), that is



Now all cars go either along BCE or BDE. Hence $x_1+x_2=6$, and

$$J_1(x_1) = 10x_1 + 50 + x_1 = 11x_1 + 50$$
$$J_2(x_2) = 50 + x_2 + 10x_2 = 11x_2 + 50$$

$$J_1(x_1) = J_2(x_2) \Rightarrow 11x_1 + 50 = 11x_2 + 50 \Rightarrow \boxed{x_1 = x_2}$$

since $x_1 + x_2 = 6 \Rightarrow \boxed{x_1^* = 3}$ $\boxed{x_2^* = 3}$

$\Rightarrow$

$J_1^*(x_1^*) = 11 \cdot 3 + 50 = 83$ time units

$J_2^*(x_2^*) = 11 \cdot 3 + 50 = 83$ " "

Conclusion: By reducing the network capacity we got reduced travelling time which represents the Braess' paradox.

If the drivers cooperate and minimize $J_1+J_2+J_3$ ~~that~~ in the original problem with path CD available (Pareto Solution) that leads to $x_1 = x_2 = 3$ and $x_3 = 0$, which imply

$J_1^P = 11 \cdot 3 + 10 \cdot 0 + 50 = 83$, $J_2^P = 11 \cdot 3 + 10 \cdot 0 + 50 = 83$

and $J_3^P = 10 \cdot 3 + 10 \cdot 3 + 10 = 70$, which is an unstable equilibrium