3.5 NASH EQUILIBRIA IN EXTENSIVE FORM

Def 3.12 Nash equilibrium (seen before)

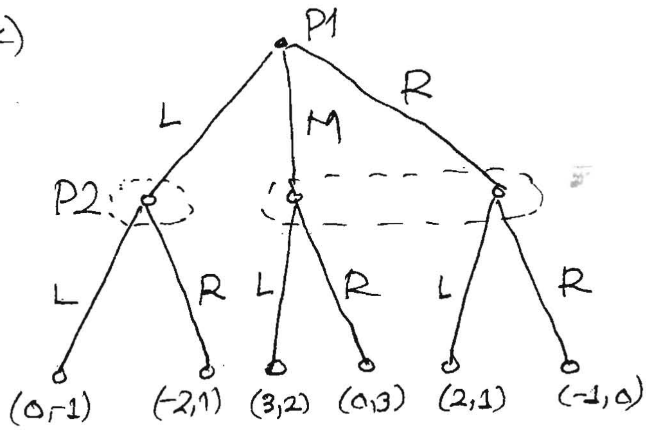
$$J_1^* = J_1(u_1^*, u_2^*, \dots, u_N^*) \leq J_1(u_1, u_2^*, u_3^*, \dots, u_N^*)$$
$$J_2^* = J_2(u_1^*, u_2^*, \dots, u_N^*) \leq J_2(u_1^*, u_2, u_3^*, \dots, u_N^*)$$
$$\vdots$$
$$J_N^* = J_N(u_1^*, u_2^*, \dots, u_N^*) \leq J_N(u_1^*, u_2^*, \dots, u_{N-1}^*, u_N)$$

3.5.1 SINGLE-ACT GAMES: PURE STRATEGY NASH EQUILIBRIA

<u>Static informational game</u>: each player has a single informational set
(they are completely equivalent to Nash games in normal form).

<u>Dynamic informational game</u>: one of the players has some information about actions of other players, hence the player has several information sets

(Ex)



(II-game)

P2 knows that P1 played either L or P1 played M or R. P2 can not distinguish between M and R strategies of P1.

We can try to analyze this game in a recursive manner, as before.

In the case $u_1 = L \Rightarrow u_2 = L \Rightarrow J_1^N = 0, \quad J_2^N = -1$

In the right-hand part (zone) of the game we have a static informational game for which the Nash equilibrium can be found by using two matrices

$$A = \begin{array}{cc} \begin{bmatrix} 3 & 0 \\ 2 & \boxed{-1} \end{bmatrix} & \begin{array}{c} M \\ R \end{array} \\ \begin{array}{cc} L & R \end{array} & \end{array}, \qquad B = \begin{array}{cc} \begin{array}{cc} L & R \end{array} & \\ \begin{bmatrix} 2 & 3 \\ 1 & \boxed{0} \end{bmatrix} & \begin{array}{c} M \\ R \end{array} \end{array}$$

no dotted lines

which implies the unique Nash equilibrium $J_1^H = -1, \quad J_2^H = 0$ obtained for $u_1 = R$ and $u_2 = R$.

It follows from this analysis that u_1 should play R which will assure for him $J_1^H = -1$. In such a case the best strategy for P2 is also R which produces $J_2^H = 0$.

However, this game is much ~~[illegible]~~ richer in strategies to be played.

The normal form of this game can be obtained as follows:

The number of pure strategies for P1 is three L, M, R

The number of pure strategies for P2, who has two information sets in which he can play either L or R is four, that is

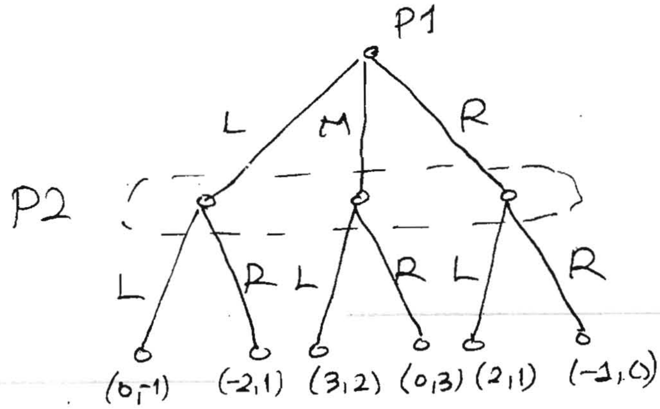
LL, LR, RL, RR

The corresponding normal game is

$$A = \begin{array}{c} \begin{array}{cccc} LL & RR & LR & RL \end{array} \\ \begin{bmatrix} \boxed{0} & -2 & 0 & -2 \\ 3 & 0 & 0 & 3 \\ 2 & -1 & \boxed{-1} & 2 \end{bmatrix} \end{array} \begin{array}{c} \\ L \\ M \\ R \end{array} \qquad B = \begin{array}{c} \begin{array}{cccc} LL & RR & LR & RL \end{array} \\ \begin{bmatrix} \boxed{-1} & 1 & -1 & 1 \\ 2 & 3 & 3 & 2 \\ 1 & 0 & \boxed{0} & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ L \\ M \\ R \end{array}$$

which reveals that this game in fact has two Nash equilibria $(0,-1)$ and $(-1,0)$, none better than the other.

The second Nash equilibrica could have been also obtained from the game's extensive form under assumption that P2 has only one information set, that is from



(I-game)

Its normal form is (now P2 has only two pure strategies L and R)

$$A = \begin{array}{c} \begin{bmatrix} \boxed{0} & -2 \\ 3 & 0 \\ 2 & -1 \end{bmatrix} \\ \begin{array}{cc} L & R \end{array} \end{array} \begin{array}{c} L \\ M \\ R \\ \\ \end{array} \qquad B = \begin{array}{c} \begin{bmatrix} \boxed{-1} & 1 \\ 2 & 3 \\ 1 & 0 \end{bmatrix} \\ \begin{array}{cc} L & R \end{array} \end{array} \begin{array}{c} L \\ M \\ R \\ \\ \end{array}$$

which implies the unique Nash equilibrium $(0,-1) = (J_1^N, J_2^N)$.
Note that game I is informationally inferior to game II

From this example we can draw some conclusions:

- existence of multiple Nash equilibria.
- recursive procedure may find only one of the Nash equilibria.
- informationally inferior game (with single information set) produces also ~~the~~ a Nash equilibrium for the original informationally superior game (with two information sets).

PROPOSITION 3.7 Let (I) be an N-person single-act game that is informationally inferior to another single-act N-person game (II). Then

(*) Any Nash equilibrium solution of (I) also constitutes a Nash equilibrium of (II)

Proof: not difficult

Hence, games with dynamic information in general admit multiple Nash equilibria (informational nonuniqueness).

To cope with the problem of informational nonuniqueness we impose more structure on the game tree and study feedback, ~~ladder,~~ nested, ladder-nested, structures. Even more, we characterize several types of Nash equilibria, such as robust equilibrium, perfect equilibrium sequential equilibria, strategic equilibria and so on.

(Ex)

$$A = B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow$$

(0,0) and (1,1) are Nash equilibria. However (1,1) is not a Nash equilibrium if elements are

$$A_\varepsilon = \begin{bmatrix} 0+\varepsilon_{11} & 1+\varepsilon_{12} \\ 1+\varepsilon_{21} & 1+\varepsilon_{22} \end{bmatrix}, \quad B_\mu = \begin{bmatrix} 0+\mu_{11} & 1+\mu_{12} \\ 1+\mu_{21} & 1+\mu_{22} \end{bmatrix}$$

We will show that there are infinitesimally small quantities ε_{ij} and μ_{ij} that destroy Nash equilibrium (1,1), hence it is not robust equilibrium

Take

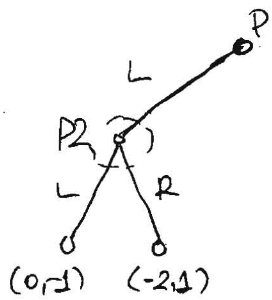
$$A_\varepsilon = \begin{bmatrix} \boxed{0.1} & 1.1 \\ 1.1 & 1.1 \end{bmatrix}, \quad B_\mu = \begin{bmatrix} \boxed{0.2} & 0.98 \\ 0.99 & 1.01 \end{bmatrix}$$

Hence, the Nash equilibrium now is only $i=1, j=1 \implies V_H = (0.1, 0.2)$

Of course, we prefer to select Nash equilibria that are robust, especially when the data in A, B matrices are obtained experimentally.

Note that in the bimatrix game at the top of page 9 the equilibrium $(0,-1)$ is not robust (if you change b_{13} to -0.99). Also, by changing b_{32} to -0.01 we see that the Nash equilibrium $(-1,0)$ is not robust

Comment: As suggested by (Ana Lucia or Andreea) after the class we can set up a normal form for the game that corresponds to the left information set, page 7



$$\iff A = \begin{bmatrix} \boxed{0} & -2 \end{bmatrix} L, \quad B = \begin{bmatrix} \boxed{-1} & 1 \end{bmatrix} L$$
$$\qquad\qquad L \quad R \qquad\qquad\qquad L \quad R$$

which reveals that $(0,-1)$ is the Nash eq

Hence, in this example, both Nash equilibria could have been detected from game's extensive form.