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FINITE NASH GAMES

(3.2) BIMATRIX GAMES

2-player finite Nash games (BIMATRIX GAMES) are non-zero sum games.

Two matrices $A^{m\times n}$ and $B^{m\times n}$ have information about costs of respectively P1 and P2

$$A^{m\times n} = \begin{array}{cc} & \quad\quad\quad \downarrow j \\ \rightarrow i & \begin{bmatrix} a_{11} & a_{12} & \cdots & & a_{1n} \\ a_{21} & a_{22} & \cdots & & a_{2n} \\ \vdots & & & & \vdots \\ a_{i1} & a_{i2} & \cdots a_{ij} & \cdots & a_{in} \\ \vdots & & & & \\ a_{m1} & a_{m2} & & & a_{mn} \end{bmatrix} \end{array}, \quad B^{m\times n} = \begin{array}{cc} & \quad\quad\quad \downarrow j \\ \rightarrow i & \begin{bmatrix} b_{11} & b_{12} & \cdots & & b_{1n} \\ b_{21} & b_{22} & \cdots & & b_{2n} \\ b_{i1} & b_{i2} & \cdots b_{ij} & \cdots & b_{in} \\ \vdots & & & & \vdots \\ b_{m1} & b_{m2} & \cdots & & b_{mn} \end{bmatrix} \end{array}$$

$$\left.\begin{array}{l} \text{P1 chooses rows} \\ \text{P2 chooses columns} \end{array}\right\} \Rightarrow (a_{ij}, b_{ij}) \text{ is the game outcome}$$

Rational behavior: each player minimizes (max) his/her losses (winnings). Positive numbers in matrix A (B) are losses for P1 (P2) and negative numbers in A(B) are gains for P1 (P2)

Def. 3.1 Nash strategies

If the pair of inequalities

$$a_{i^*j^*} \le a_{ij^*}$$

$$b_{i^*j^*} \le b_{i^*j}$$

is satisfied for all $i=1,2,..,m$ and $j=1,2,..,n$

The pair $(a_{i^*j^*}, b_{i^*j^*})$ is the Nash equilibrium outcome of the bimatrix game.

In general, as given in Def. 3-1 Nash equilibria are ill-defined as demonstrated in the next examples.

(Example) $A = \begin{bmatrix} \boxed{1} & 0 \\ 2 & \boxed{-1} \end{bmatrix}$, $B = \begin{bmatrix} \boxed{2} & 3 \\ 1 & \boxed{0} \end{bmatrix}$

P1 examines the columns for every j-strategy of P2 and chooses the smallest element as indicated by dashed lines.

P2 examines the rows for every i-strategy of P1 and chooses the smallest element in the rows, as indicated by dashed lines.

In this game we have two Nash equilibria that satisfy Definition 3.1 (solid-line boxes)

$(i=1, j=1)$ and $(i=2, j=2)$

$\Downarrow$ $\Downarrow$

$(1,2)$ = game outcomes = $(-1,0)$

Apparently, the players without any need for cooperation will choose $(i=2, j=2)$ since this strategy provides a better Nash equilibrium for both players.

Definition 3.2 A pair of strategies (i_1, j_1) is better than another pair (i_2, j_2) if

$a_{i_1 j_1} \leq a_{i_2 j_2}$

$b_{i_1 j_1} \leq b_{i_2 j_2}$

with at least one strict inequality.

Definition 3.2 A Nash equilibrium is admissible if there exists no better Nash equilibrium.

Note that ordering in the pairs on numbers is not a complete operation since

$(1, 2) < (3, 4)$

but

$(1, 2)\ ?\ (2, 1)$

(Example)

$$A = \begin{bmatrix} \boxed{-2} & 1 \\ +1 & \boxed{-1} \end{bmatrix}, \quad B = \begin{bmatrix} \boxed{-1} & 1 \\ 2 & \boxed{-2} \end{bmatrix}$$

$\Rightarrow$ two Nash equilibria

$(i=1, j=1) \Rightarrow (-2,-1)$ — can not be claimed which one is better

$(i=2, j=2) \Rightarrow (-1,-2)$

However, the result of the game ~~is not~~ might not be any of the Nash equilibria since P1 looking at the entries in A may choose $i=1$ and P2 may choose $j=2$ $\Rightarrow$ $(1,1)$ as a game outcome which is apparently worse for both players

$$(-1,-2) < (1,1)$$
$$(-2,-1) < (1,1)$$

This is a serious problem with pure Nash strategies since the ~~that~~ Nash equilibria are not recognized by the players $\Rightarrow$ the outcome that may be worse for both players.

This leads to a conclusion that in the space of pure strategies, in the case of multiple Nash equilibria, the Nash equilibrium is not well-defined (unless we allow communications (cooperation), which is not the rule of the game).

(Example)

$$A = \begin{bmatrix} \boxed{8} & 0 \\ 30 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} \boxed{8} & 30 \\ 0 & 2 \end{bmatrix}$$

Here, $i=1$, $j=1$ imply the unique Nash equilibrium. The game is well-posed. The players play $i=1$, $j=1$. The outcome of the game is Nash equilibrium equal to 8.

SECURED STRATEGIES and MINIMAX SOLUTION

We use the same logic as before.

<u>The secured strategy for P1:</u>
the row in A whose maximal element is minimal (the row with the minimal maximal element $\min_i \max_j \{a_{ij}\}$)

<u>The secured strategy for P2:</u>
The column in B with the minimal maximal element $(\min_j \max_i \{a_{ij}\})$

(Ex)

$$A = \begin{array}{c l} \begin{bmatrix} 1 & \diamond{-2} \\ 3 & 2 \end{bmatrix} & \leftarrow i=1 \end{array} , \quad B = \begin{array}{c} \begin{bmatrix} 2 & \diamond{-1} \\ 1 & 0 \end{bmatrix} \\ \uparrow j=2 \end{array}$$

( ) = maximal row (column) element

$\diamond$ = the value of game (security strategies played)

Since both players use min max to find their secured strategies, the corresponding solution is known as <u>minmax solution</u>.

Note that in this example the security strategies coincide with Nash strategies since $(-2,-1)$ is the unique Nash equilibri[um] also

$$A = \begin{bmatrix} 1 & \boxed{-2} \\ 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & \boxed{-1} \\ 1 & (0) \end{bmatrix}$$

$\Rightarrow V_{Nash} = (-2,-1)$ for $i=1$ and $j=2$

$V_{Nash} = V_{secure} = V_s$

In general the minmax solution is "worse" than the Nash solution as demonstrated on the next example

$$A_N = \begin{bmatrix} 1 & -2 \\ -3 & 2 \end{bmatrix} \leftarrow i_N \quad B_N = \begin{array}{cc} \downarrow j_N & \\ \begin{bmatrix} -1 & 1 \\ \boxed{-1} & 0 \end{bmatrix} \end{array} \Rightarrow V_N = (-3, -1)$$

$i_N = 2,\ j_N = 1$

$$A_S = \begin{bmatrix} 1 & -2 \\ -3 & 2 \end{bmatrix} \leftarrow i_S, \quad B_S = \begin{array}{cc} \downarrow j_S & \\ \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \end{array} \Rightarrow V_S = (1, -1)$$

$i_S = 1,\ j_S = 1$

MIXED STRATEGIES

(Ex)

no double-lines

$$A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$$

no Nash equilibrium here in pure strategies.

<u>Definition 3.6</u> Mixed Nash strategies

A pair $\{y^* \in Y = \{y_i : y_i \geq 0, \sum_{i=1}^{m} y_i = 1\},$

$z^* \in Z = \{z_j : z_j \geq 0, \sum_{j=1}^{n} z_j = 1\}$

is a Nash equilibrium solution to a bimatrix game in mixed strategies if

$$y^{*T} A z^* \leq y^T A z^*, \quad y \in Y$$

$$y^{*T} B z^* \leq y^{*T} B z, \quad z \in Z$$

the pair $(y^{*T} A z^*, y^{*T} B z^*)$ is the game value of the Nash equilibrium

<u>Theorem 3.1</u> <u>Every bimatrix game has at least one Nash equilibrium in mixed strategies.</u>

Computation of Nash equilibrium in mixed strategies is pretty difficult.

3.3 N-Person Finite Nash Games in Normal Form

- N-players P1, P2, ..., PN
- Each player has a finite number of strategies, m_i, with n_i denoting a strategy of the player i.
- For given n_i's the cost functions of each player are

$$J^i = a^i_{n_1 n_2 n_3 \dots n_N}, \quad i = 1, 2, \dots, N$$

- Rule of the game: each player minimizes his/her cost function independently (assuming that the other players are doing the same).

Def. 3.7 Nash Equilibrium

An N-tuple of strategies $(n_1^*, n_2^*, \dots, n_N^*)$ is a Nash equilibrium if the following N-inequalities are satisfied

$$J^{1*} = a^1_{n_1^* n_2^* \dots n_N^*} \le a^1_{n_1 n_2^* \dots n_N^*} \quad (1)$$

$$J^{2*} = a^2_{n_1^* n_2^* n_3^* \dots n_N^*} \le a^2_{n_1^* n_2 n_3^* \dots n_N^*} \quad (2)$$

$$\vdots$$

$$J^{N*} = a^N_{n_1^* n_2^* \dots n_{N-1}^* n_N^*} \le a^N_{n_1^* n_2^* \dots n_{N-1}^* n_N} \quad (N)$$

$\boxed{\text{Theorem } 3.2}$ Every N-person static finite game in normal form admits ~~that~~ at least one Nash equilibrium in mixed strategies. (no proof)

In general, Nash equilibria in normal form are not unique, difficult to find (calculate) and lead to ill-posedness of the corresponding game.