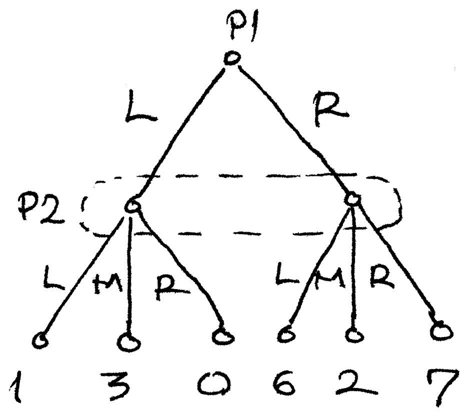
2.4 EXTENSIVE FORM

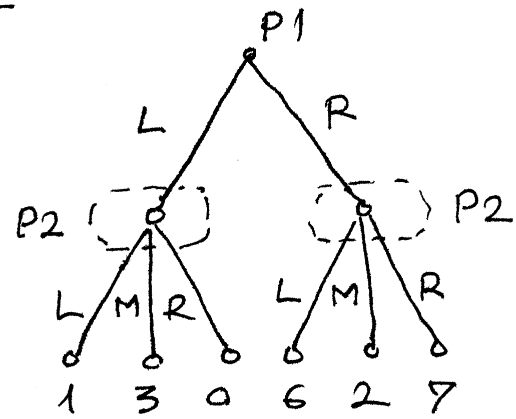
Feb. 12, 99 (4)

It is given by a tree structure with explicit description of the order of play and information available to each player.

Information structure: (Examples)



(a)



(b)

The numbers at the end points indicate losses of P1.

Examples (a) and (b) have different information structure. In the case of (a) P2 does not have information about the chosen strategy of P1, hence P2 does not know at what node (left or right) the game is progressed. This corresponds to a normal form with simultaneous decision making, which can be represented by the following matrix

$$(a) \Longrightarrow \quad P1 \begin{array}{c|ccc} & & P2 & \\ & L & M & R \\ \hline L & 1 & 3 & 0 \\ R & 6 & 2 & 7 \end{array}$$

$$\Longrightarrow \quad \bar{V} = 3 \ (\gamma_1 = L)$$

$$\underline{V} = 2 \ (\delta_2 = M)$$

$$V_m = \frac{8}{3} \left(\begin{array}{l} y_L = \frac{2}{3},\ y_R = \frac{1}{3} \\ z_L = \frac{1}{2},\ z_M = \frac{2}{3} \end{array} \right.$$

In such a case ((a)) the number of strategies for all nodes within the dashed-contour must be the same and P2 chooses one of them (for example M, which in the case when P1 plays L ⇒ 3 as the game result, and for P1 played R ⇒ 2 is the game result.

<u>Case (b)</u>, indicates different information in the left and right nodes for P2. The number of strategies for P2 at the left and right nodes <u>are not necessarily the same</u>. In this case P2 can choose any of the strategies (L, L), (L, M), (L, R), (M, L), ... and so on (R, R), for the total of nine strategies for P2. P1 has only two available strategies L ~~and~~ R so that the corresponding matrix game is given by

P2

		(L,L)	(L,M)	(L,R)	(M,L)	(M,M)	(M,R)	(R,L)	(R,M)	(R,R)	
P1	L	1	1	1	3	3	3	0	0	0	← $i=1$
	R	6	2	7	6	2	7	6	2	7	
					↑ $j=4$		↑ $j=6$				

$$\Rightarrow \left.\begin{aligned} \overline{V} &= 3 \\ \underline{V} &= 3 \end{aligned}\right\} \Rightarrow$$

$(i=1, j=4)$ and $(i=1, j=6)$ are the equilibrium (saddle point strategies with the game value

$$V = \overline{V} = \underline{V} = 3$$

We conclude that (a) and (b) are two completely different games. (a) has no equilibrium in pure strategies. On the other hand, (b) has two equilibria in pure strategies.

The original definition of a finite game given on page 6, of handout (1) now can be expanded to include the information structure

1) A finite tree - -
2) Utility loss functions assigned to terminal nodes - .
3) Partitions of internal nodes S_1, S_2. indicating the order of play
4) Partitions of S_i into S_i^j according to different information structures.

(For a zero-sum finite game the corresponding definition is given on pages 38-39, Definition 2.5)

Let $\gamma^1 \in \Gamma^1$ and $\gamma^2 \in \Gamma^2$ denote strategies of P1 and P2.

Definition 2.8 (Saddle-Point)

$$J(\gamma^{1*}, \gamma^2) \leq J(\gamma^{1*}, \gamma^{2*}) \leq J(\gamma^1, \gamma^{2*})$$

$J(\gamma^{1*}, \gamma^{2*})$ = the saddle point values of the game

It follows that the game (b) is represented in the normal form by the matrix $A^{2\times 9}$. In the case when P2 has more strategies the order of the matrix might be very high.

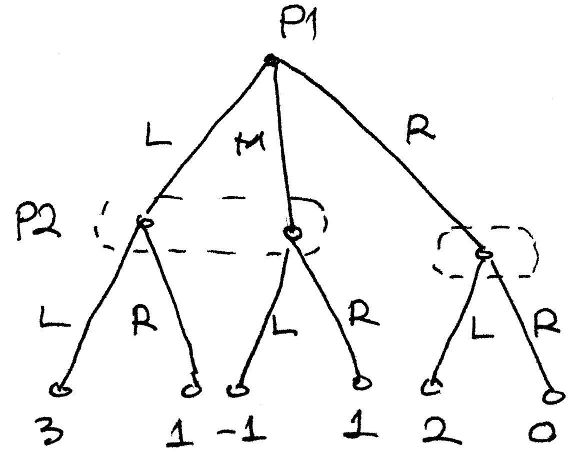
Another approach is based on a recursive procedure

Note that by examining the game ~~be~~ (b), following the tree, it can be easily concluded that

$$\gamma^{1*} = L \qquad \Rightarrow \qquad \gamma^{*2} = (M, R)$$

in the case that P1 does not use his optimal strategy

(Example)

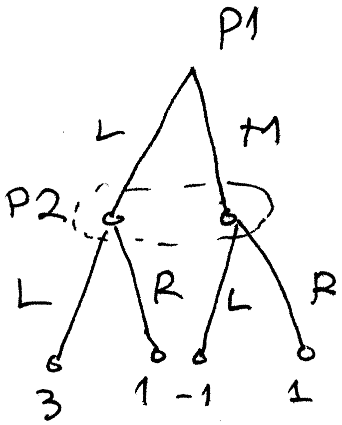


my notation — book's notation

$u_1 = \delta^1$
$u_2 = \delta^2$
$J = V$

It is obvious that for $u_1 = R$, $u_2(u_1 = R) = L \Rightarrow \underline{V = 2}$

However, for $u_1 = L$ or $u_1 = M$ we have another independent game with simultaneous decision making



$\Leftrightarrow$

P1 \ P2	L	R
L	3	1
M	-1	1

$\leftarrow i = 2$

$\uparrow j = 2$

$\Downarrow$

$\bar{V} = 1, \ \underline{V} = 1$

Hence, we have here the unique saddle-point obtained for $u_1 = M$ $(i = 2)$ and $u_2 = R$ $(j = 2) \Rightarrow \underline{V = 1}$

By solving both problems with $(V = 2 \Leftarrow u_1 = R)$ and $(V = 1 \Leftarrow u_1 = M)$, P1 concludes that his optimal strategy is

$$u_1^* = M$$

which imply

$$u_2^*(M) = R \Rightarrow V = 1^*$$

In addition, P2 must be ready for other choices of P1, that is

$u_2^*(R) = L \Rightarrow V = 2$

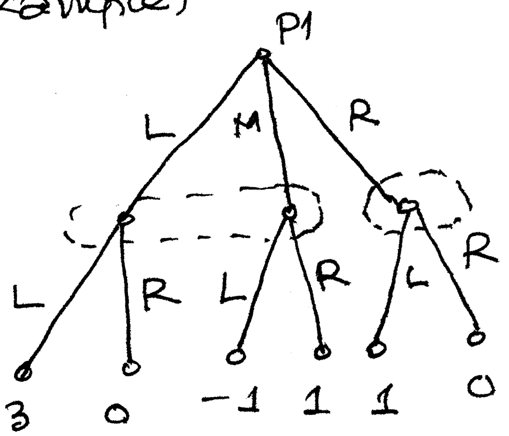
$u_2^*(L) = R \Rightarrow V = 1$

Algorithm for Solving Zero-Sum Games in Extensive Form

1) For each information set of P2 solve the corresponding matrix game (assuming that each of them has a saddle-point in pure strategies).
2) P1 chooses the path to the smallest value of the saddle-points found in (1).
3) P2 has to chose the saddle point strategies in all matrix games considered in 1.

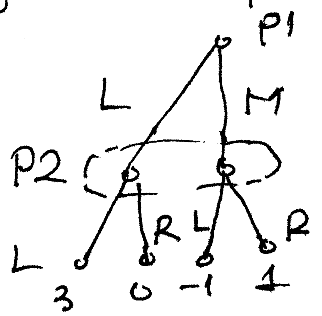
If one of the saddle points does not exist, we have to use the <u>mixed</u> strategies

(Example)



If $u_1 = R \Rightarrow u_2 = L \Rightarrow J = 1 = V$

If $u_1 = \{L \text{ or } M\}$ we have to form the matrix game for this part of the tree (zone)



$\Longrightarrow$ P1

		P2	
		L	R
P1	L	3	0
	M	−1	1

$\Longrightarrow$ $\bar{V} = 1 \quad i = 2$

$\underline{V} = 0 \quad j = 1$

no saddle-point in pure strategies

However, this game has the saddle point in mixed strategies (see, page 29)

$y_L^* = 2/5, \ y_M^* = \frac{3}{5}, \ z_L^* = \frac{1}{5}, \ z_R^* = \frac{4}{5}, \ V_m = 3/5$

Hence

$$\hat{u}_1^* = \begin{cases} L & \text{w.p. } 2/5 \\ M & \text{w.p. } 3/5 \\ R & \ldots \end{cases}$$

$$\hat{u}_2^* = \begin{cases} L & \text{w.p. } 1 \quad \text{if } u_1 = R \\ L & \text{w.p. } 1/5 \\ R & \ldots \end{cases} \text{ if } u_1 = u_1^*$$

$\hat{u}_1^*$ and $\hat{u}_2^*$ are known as behavioral strategies.

Note that P1 has three possible pure strategies

$$u_1 = \begin{cases} L \\ M \\ R \end{cases}$$ hence $\hat{u}_1^*$ is its optimal mixed strategy

However, P2 has four pure strategies

$u_2(u_1=R)=L$
$u_2(u_1=R)=R$

$u_2(u_1=L \text{ or } M)=L$

$u_2(u_1=L \text{ or } M)=R$

on the ~~right-hand~~ part of the game P2 has two alternatives L and R.
Also on the left-hand part of the game P2 has two alternatives L and R.
The total number of his pure strategies is four
(L,L), (L,R), (R,L) and (R,R)

Hence, due to the fact that P2 has four pure strategies the probability distribution should be assigned to them.
However, there is no reason for P2 to play R in the right-hand half of the game since he knows the game's behavior for $u_1=R \Rightarrow$
and there is no need to P2 to play $u_2=R$ in that part of the game. Thus, the pure strategies (L,R) and (R,R) can be discarded by P2 and he only needs to assign probabilities to (L,L) and (R,L) pure strategies.

Hence $\hat{\gamma}^{2*} = u_2^* = \begin{cases} (L,L) & \text{w.p. } 1/5 \\ (L,R) & \text{w.p. } 4/5 \end{cases}$

book's notation

Remark 2.5 Every behavioral strategy is a mixed strategy.

Definition 2.10 Saddle-point in behavioral strategies

$$\hat{J}(\hat{u}_1^*, u_2) \le \hat{J}(\hat{u}_1^*, \hat{u}_2^*) \le \hat{J}(\hat{u}_1, \hat{u}_2^*)$$

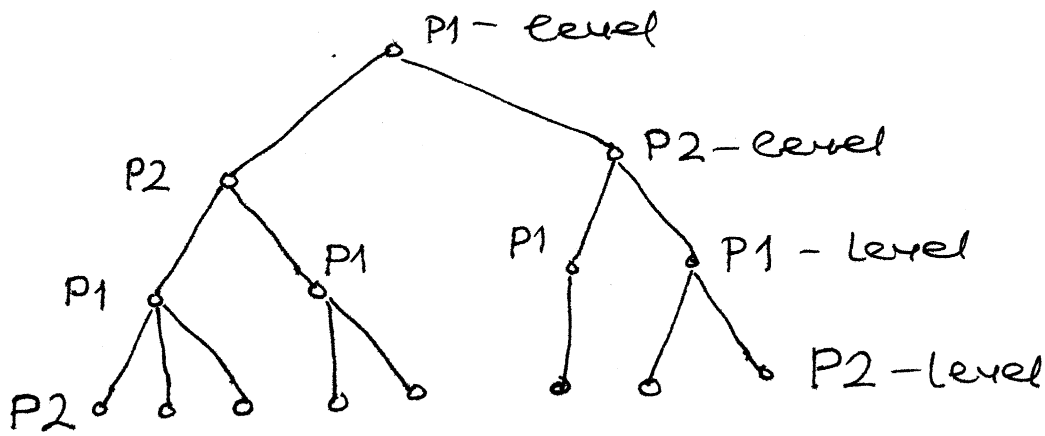
Corollary 2.4 In a single-act (level) two person zero-sum game:

(i) there exists at least one saddle point in behavioral strategies.

(ii) the saddle-point of (i) is also the saddle point in mixed strategies.

2.5 MULTI-ACT (LEVEL) GAMES

(Ex)

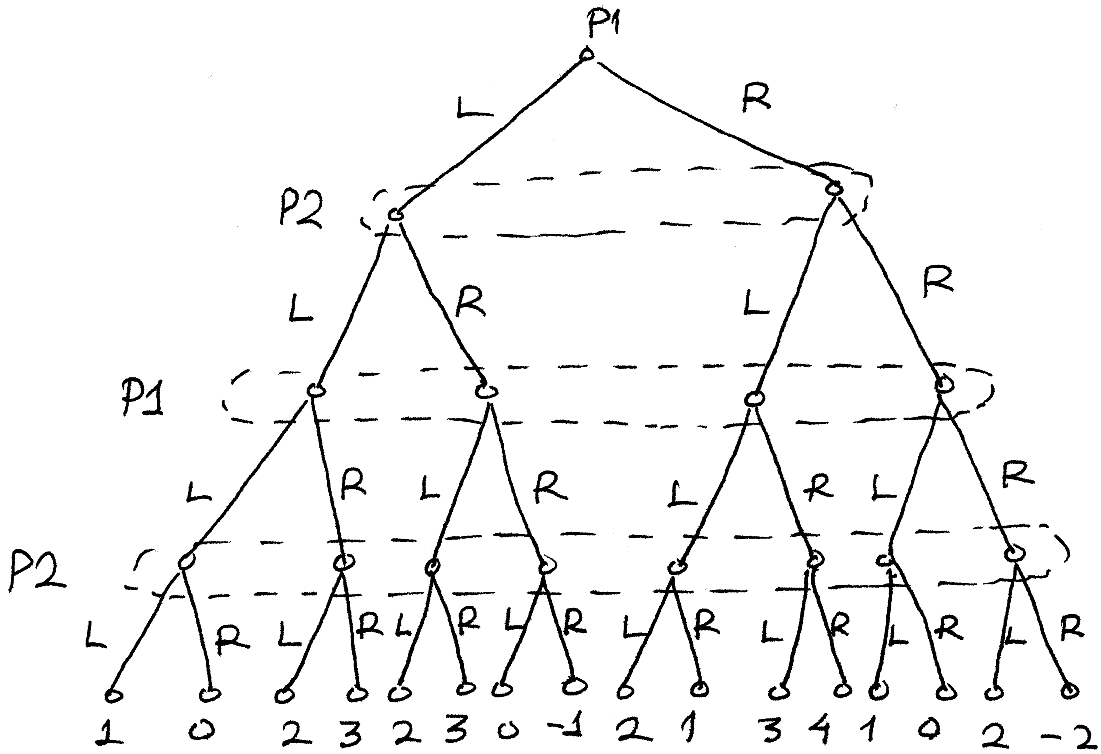


It gets pretty complex in general.

Two special classes of multi-level games in extensive form are the open-loop (no information) and the closed-loop (full information) games

Def 2.13 A multi-act game is said to be an open-loop game if at each level of play each player has a single information set.

(Example 2.4) An open-loop game



Note that each player has twice to make a decision L or R, hence the total number of strategies for each player is four

LL, LR, RL, and RR

which can be recorded in a normal form of

P2

P1	LL	LR	RL	RR
LL	1	0	2	3
LR	2	3	0	-1
RL	2	1	1	0
RR	3	4	2	-2

← $i=3 \Rightarrow \bar{V}=2$

↑ $j=1 \Rightarrow \underline{V}=1$

$\underline{V} \neq \bar{V}$ no saddle-point in pure strategies

However, it can be shown that

$\hat{u}_1^* = \begin{cases} LL & \text{w.p. } 3/5 \\ RR & \text{w.p. } 2/5 \end{cases}$ and $\hat{u}_2^* = \begin{cases} LL & \text{w.p } 4/5 \\ RR & \text{w.p } 1/5 \end{cases}$

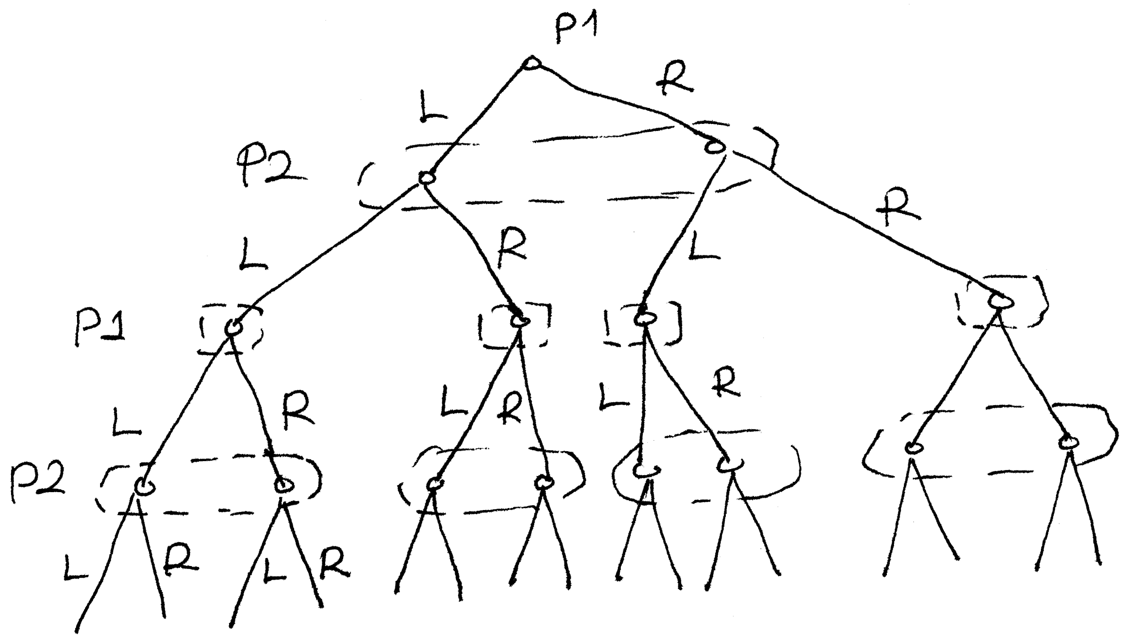
induces the saddle point solution with $\underline{V} < V_m = \frac{7}{5} < \bar{V}$.

$\boxed{\text{Def 2-11}}$ A multi-act game is said to be a *feedback game* if

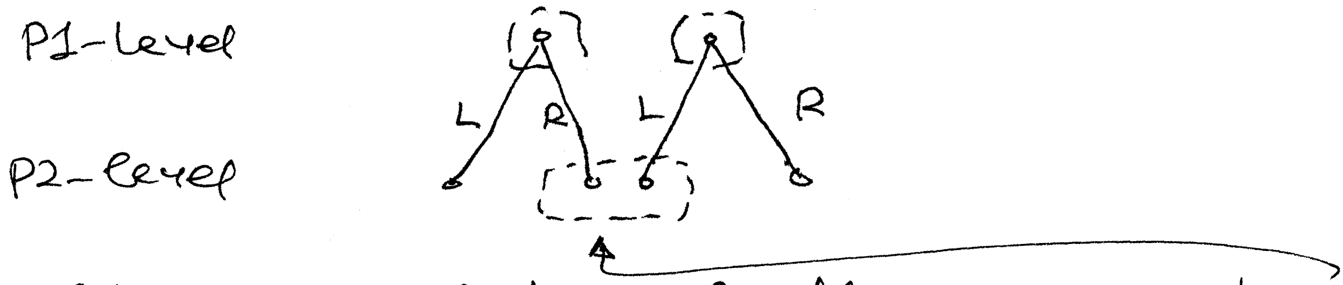
(i) no information set contains nodes from different levels of play.

(ii) information sets for P1 are singletons (contain only one node) at every level of play, and information sets for P2 do not contain nodes corresponding to the branches coming from different information sets

(Example of a feedback game)



If we have



this can not be a feedback game because of the information set for P2 is obtained from different information sets of P1

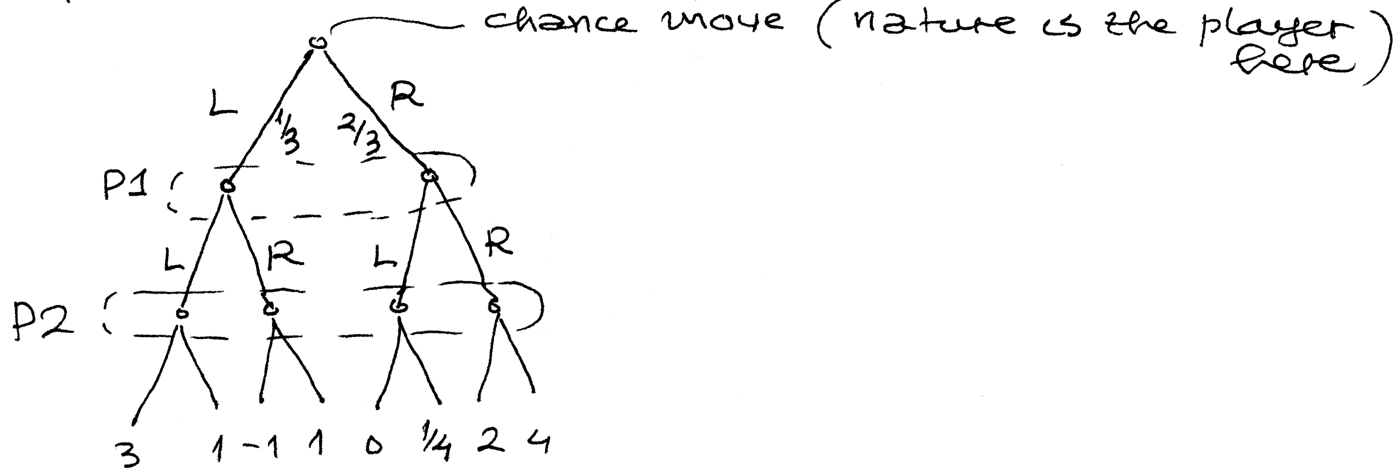
Some facts:

(Proposition 2.4) Every zero-sum finite multi-act feedback game admits a saddle-point solution in behavioral strategies.

(Proposition 2.5) Every zero-sum finite multi-act game admits a saddle-point solution in mixed strategies.

2.6 ZERO-SUM GAMES WITH CHANCE MOVES

(Example)



The rest is the same as before

Comment on the Proof of the Minimax Theorem

In order to fill a gap in the proof, as suggested by one of the students we use:

Let $\tilde{A}$ is an arbitrary matrix such that

$$\tilde{A} = A - \begin{bmatrix} c & \cdots & c \\ \vdots & & \vdots \\ c & \cdots & c \end{bmatrix}, \quad A \text{ - is a game matrix, } c = \text{constant}$$

Then by Lemma 2.1

$$y^T \tilde{A} \le 0 \quad \Rightarrow y^T \tilde{A} z \le 0 \Rightarrow y^T \left(A - \begin{bmatrix} c & \cdots & c \\ \vdots & & \vdots \\ c & \cdots & c \end{bmatrix}\right) z \le 0$$

$$\Rightarrow y^T A z \le y^T \begin{bmatrix} c & \cdots & c \\ \vdots & & \vdots \\ c & \cdots & c \end{bmatrix} z = \left(\sum_{i=1}^{m} y_i\right) c \left(\sum_{j=1}^{n} z_j\right) = 1 \cdot c \cdot 1 = c$$