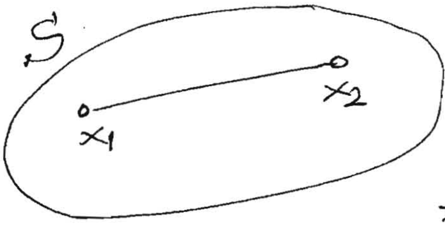
MATH BACKGOUND – CONVEX SETS

Convex Set : If the line segment joining any two points of the set also belong to the set



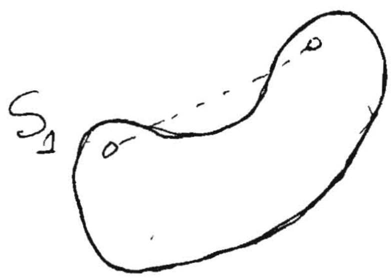
$$x_1, x_2 \in S$$

$$x = \lambda_1 x_1 + \lambda_2 x_2 \in S \qquad \lambda_1 + \lambda_2 = 1$$
$$\lambda_1, \lambda_2 \geq 0$$

or

$$\left(\lambda x_1 + (1-\lambda) x_2 \in S \right), \; 0 \leq \lambda \leq 1$$

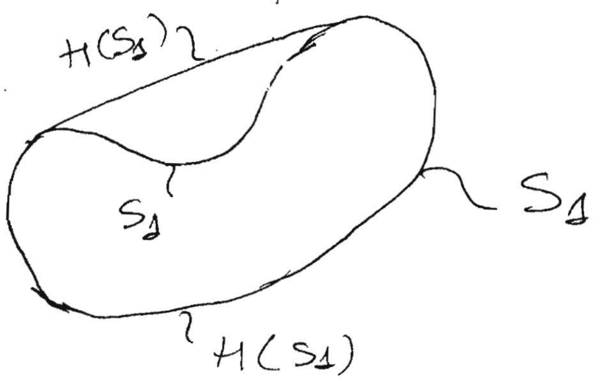
x = convex combination of x_1 and x_2.



This is a nonconvex set.

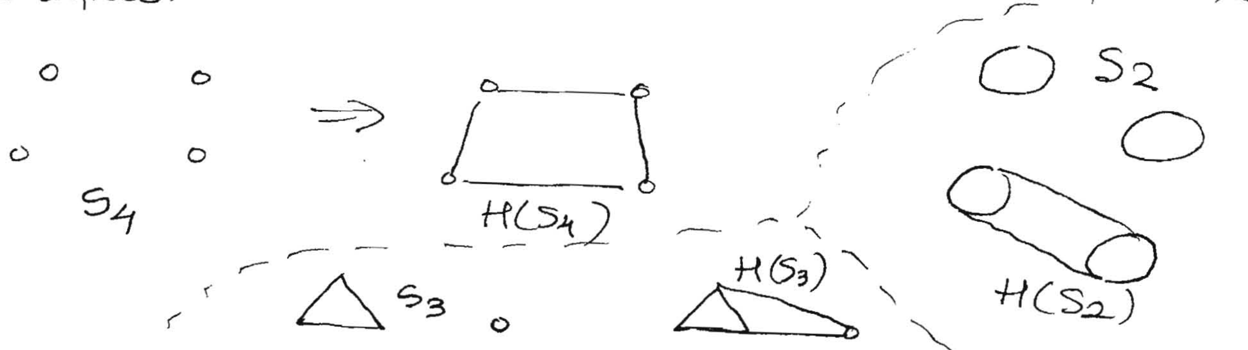
Convex hull: $H(S_1)$ is a minimal convex set that contains S_1.

For example



Note that S_1 is an arbitrary set

(Examples)

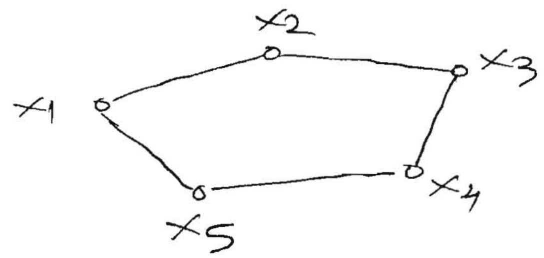


It holds for the convex hull (and for the convex set)

$$x \in H(S) \quad \text{iff} \quad x = \sum_{j=1}^{k} \lambda_j x_j \quad , \; \lambda_j \geq 0 \, , \; \sum_{j=1}^{k} \lambda_j = 1$$

Polytope: A convex hull of a finite number of points $x_1, \ldots, x_{k+1}$ in E_n is called a polytope

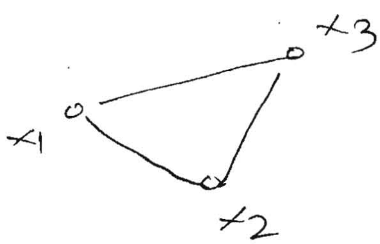
(Ex)



= polytope

Simplex: Let $x_1, x_2, \ldots, x_{k+1}$ form a polytope in E^n.
If $x_2 - x_1, x_3 - x_1, \ldots, x_{k+1} - x_1$ are k linearly independent vectors then $H(x_1, x_2, \ldots, x_{k+1})$ defines a simplex.

In plane $(n=2)$



is a simplex

In E_n space a simplex has no more than $n+1$ vertices.

Hyperplane: $S = \{x : p^T x = \alpha\}$ defines a hyperplane

$x, p \in \mathbb{R}^n$ $\quad$ p = normal to hyperplane, α = scalar

Half space

$$S = \{x : p^T x \leq \alpha\} \quad , \; x \in E^n , \; p \in E^n$$

MINIMAX Theorem (Theorem 2.4, p.27)

In any matrix game A, the average security levels of the players in mixed strategies coincide, that is

$$\bar{V}_m(A) = \min_{Y} \max_{Z} \{y^T A z\} = \max_{Z} \min_{Y} \{y^T A z\} = \underline{V}_m(A)$$

where

$$Y = \{y \in \mathbb{R}^m;\ y_i \geq 0,\ \sum_{i=1}^{m} y_i = 1\}$$

$$Z = \{z \in \mathbb{R}^n;\ z_i \geq 0,\ \sum_{j=1}^{n} z_j = 1\}$$

Proof: We first need to prove the following lemma.

Lemma 2.1 Let A be an arbitrary $(m \times n)$-dimensional matrix. Then, either there exists

(i) a nonzero vector $y \in \mathbb{R}^m$, $y \geq 0$, such that $A^T y \leq \underline{0}$

(ii) a nonzero vector $z \in \mathbb{R}^n$, $z \geq 0$, such that $Az \geq \underline{0}$

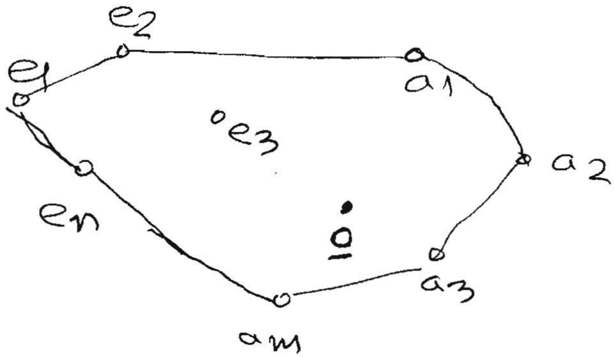
Note that $\underline{0} \in \mathbb{R}^n$ denotes a zero vector

Proof of Lemma 2.1

Let $H(e_1, e_2, \ldots, e_n, a_1', a_2', \ldots, a_m')$ denote the convex hull of $n+m$ vectors

e_i = unit vectors in E^n

a_i = rows of $A^{m \times n}$ $\Longrightarrow$ $a_i \in E^n$



A vector $\underline{o}$ either belongs to H or does not belong to H.

<u>Case $\underline{o} \in H$</u> $\Rightarrow$ A set of scalars y_i and η_j exists such that

$$\underline{o} = \sum_{i=1}^{m} y_i a_i^T + \sum_{j=1}^{n} \eta_j e_j \quad \text{, with } \sum_{i=1}^{m} y_i + \sum_{j=1}^{n} \eta_j = 1$$

$$y_i \geq 0, \quad \eta_j \geq 0$$

or in the scalar form (for every component of $\underline{o}$)

$$0 = \sum_{i=1}^{m} y_i a_{ij} + \eta_j$$

which implies

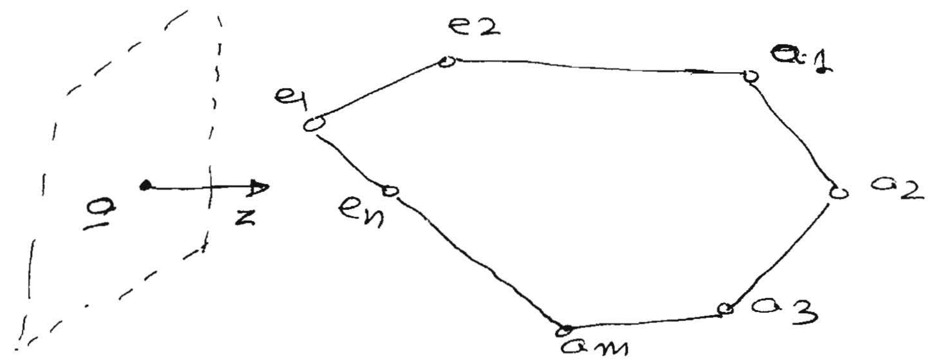
$$\sum_{i=1}^{m} y_i a_{ij} = 0 - \eta_j \leq 0 \qquad \text{since } \eta_j \geq 0$$

$$j = 1, 2, \ldots, n$$

Note

$$\sum_{i=1}^{m} y_i a_{ij} \leq 0, \; j = 1, 2, \ldots, n \iff A^T y \leq \underline{c}$$

<u>Case $\underline{o} \notin H$</u>



Now there exists a hyperplane through the $\underline{o}$ such that

$$z^T x = 0 \quad \text{and} \quad z^T x \geq 0 \quad \text{for } x \in H$$

$z^{+}x \geq 0$ for $x \in H$

Let $x = e_i$ $\Rightarrow z^T e_i = z_i \geq 0$

Let $x = a_i^T$ $\Rightarrow z^T a_i^T \geq 0 \Leftrightarrow Az \geq 0$ q.e.d

Proof of Theorem 2.4

$$\bar{V}_m(A) = \min_Y \max_Z \{y^T A z\} = \max_Z \min_Y \{y^T A z\} = \underline{V}_m(A)$$

We have established before that $\underline{V}_m(A) \leq \bar{V}_m(A)$

From Lemma 2.1 it exist $y^\circ \geq 0$ such that $A^T y^\circ \leq 0$

Also

$$y^{\circ T} A \leq 0$$

since $z \geq 0 \Rightarrow y^{\circ T} A z \leq 0$

also

$$\max_Z \{y^{\circ T} A z\} \leq 0$$

since $y^\circ \geq 0$ we have

$$\bar{V}_m(A) \min_Y \max_Z \{y^T A z\} \leq 0 \qquad (1)$$

Similarly, the second alternative of Lemma 2.1 implies that it exists z° vector such that $Az^\circ \geq 0$

or

$$y^T A z^\circ \geq 0 \quad \text{since } y \geq 0$$

or

$$\min_Y \{y^T A z^\circ\} \geq 0$$

or

$$\underline{V}_m(A) = \max_Z \min_Y \{y^T A z\} \geq 0 \qquad (2)$$

(1) and (2) were obtained for an arbitrary matrix A. If we replace this matrix by a matrix obtained by shifting all entries by a constant c, that is $a_{ij} - c$, then the game lower and upper values in mixed strategies are

$$\bar{V}_m(A) - c \quad \text{and} \quad \underline{V}_m(A) - c$$

Combining this observation with (1) and (2) we have

$$\bar{V}_m(A) \le c \qquad (3)$$

$$\underline{V}_m(A) \ge c \qquad (4)$$

At least one ~~of~~ of these inequalities must hold for an arbitrary constant c. In addition, we know that

$$\underline{V}_m(A) \le \bar{V}_m(A)$$

which implies

$$\boxed{\underline{V}_m(A) = \bar{V}_m(A)}$$

Corollary 2.3 (of Theorem 2.4)

(–) The game has a saddle point in mixed strategies.

(–) the equilibrium strategies are the mixed security strategies.

(–) $V_m(A) = \underline{V}_m(A) = \bar{V}_m(A)$

Comments on the proof of Minimax Theorem (12a)

Owen, "Game Theory" Saunders 1968 (Academic Press, 1982).

"This theorem, the most important of game theory ..."

(can be found in Owen's book)

Basar and Olsder, pp. 75, "the original proof of the minimax theorem given by Von Neumann is nonelementary and rather complicated. ... The proof given here seems to be the simplest one available in the literature.

Additional comments on the proof from the textbook

$$\left.\begin{array}{ll}(3) & \bar{V}_m(A) \le c \\ (4) & \underline{V}_m(A) \ge c\end{array}\right\}$$ one of them holds for an arbitrary constant c

We also know that

lower game value in mixed strategies $= \underline{V}_m(A) \le \bar{V}_m(A) =$ upper game value in mixed strategies

Let $\bar{V}_m(A) = \underline{V}_m(A) + k$, for some $k \ge 0$ (5)

Let us choose the constant c as

$$c = \underline{V}_m(A) + \frac{1}{2}k$$

then

(3) $\bar{V}_m(A) \le \underline{V}_m + \frac{1}{2}k = \bar{V}_m(A) - k + \frac{1}{2}k \Rightarrow 0 \le -\frac{1}{2}k$
$\Rightarrow k \le 0$

(4) $\underline{V}_m(A) \ge \underline{V}_m(A) + \frac{1}{2}k \Rightarrow 0 \ge \frac{1}{2}k \Rightarrow k \le 0$

Hence, $\underbrace{\text{(3) and (4)}}_{k \le 0}$ contradict $\underbrace{\text{(5)}}_{k \ge 0}$ unless $k = 0$

which implies $\bar{V}_m(A) = \underline{V}_m(A)$.

23 Computation of Mixed Strategies

a) Graphical Approach (for 2×2, $2\times n$, $n\times2$, games)

Example:

$$P1 \quad \overset{P2}{\begin{array}{|c|c|} \hline 3 & 0 \\ \hline -1 & 1 \\ \hline \end{array}} \quad \Rightarrow \underline{V} = 0 \;, \quad \bar{V} = 1$$

$$V_m = \underline{V}_m = \bar{V}_m = ?$$

To find the mixed strategy of P1 we assume that P2 plays only pure strategies, that is

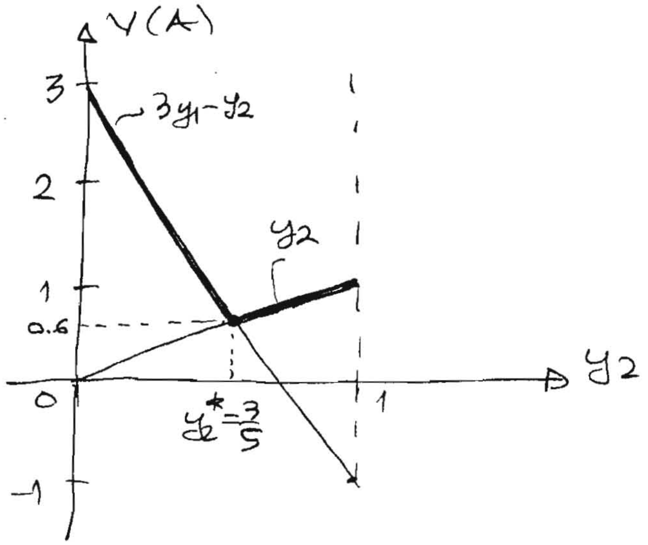
$(z_1 = 1 \,,\; z_2 = 0)$

or $(z_1 = 0 \,,\; z_2 = 1)$ $\qquad z_1 + z_2 = 1 \,,\quad z_1 \geq 0 \,,\quad z_2 \geq 0$

The problem is to find (y_1, y_2), $y_1 \geq 0, y_2 \geq 0, y_1 + y_2 = 1$

For $z_1 = 1, z_2 = 0$ we have

$$V(A) = (y_1 \; y_2)\begin{pmatrix} 3 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 3y_1 - y_2 = V(A)$$

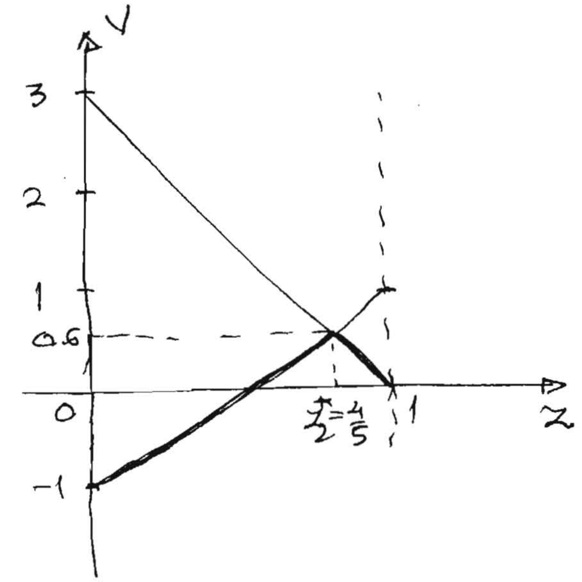


solid lines determine the average outcome of the game for different choices of mixed strategies for P1

For $(z_1 = 0, z_2 = 1)$ we have $\quad V(A) = (y_1 \; y_2)\begin{pmatrix} 3 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = y_2$

$$\left.\begin{array}{l} 3y_1 - y_2 = y_2 \\ y_1 + y_2 = 1 \end{array}\right\} \Rightarrow y_1^* = \frac{2}{5} \,,\; y_2^* = \frac{3}{5} \quad \Rightarrow \underline{V}_m(A) = 0.6$$

Similarly, for P2 we get



for $(y_1=1,\ y_2=0)$

$$V=(1\ \ 0)\begin{pmatrix}3&0\\-1&1\end{pmatrix}\begin{pmatrix}z_1\\z_2\end{pmatrix}=(3\ \ 0)\begin{pmatrix}z_1\\z_2\end{pmatrix}=3z_1$$

for $(y_1=0,\ y_2=1)$

$$V=(0\ \ 1)\begin{pmatrix}3&0\\-1&1\end{pmatrix}\begin{pmatrix}z_1\\z_2\end{pmatrix}=(-1\ \ 1)\begin{pmatrix}z_1\\z_2\end{pmatrix}=-z_1+z_2$$

$V_m=0.6$ is guaranteed for P2 if he plays his mixed security strategy $(z_1^*=\frac{1}{5},\ z_2^*=\frac{4}{5})$

For higher order game problems $n\geq 3$ and $m\geq 3$ we use the linear programming

Zero-sum finite normal game $\Longleftrightarrow$ Linear Programming problem

Matrix games can be simplified by eliminating non dominant rows and columns, if any.

Row i strictly dominates row k if $a_{ij}\leq a_{kj}$, $j=1,2,\dots,n$ and if at least for one j the strict inequality holds.

Column j strictly dominates column ℓ if

$$a_{ij}\geq a_{i\ell},\quad i=1,2,\dots,m$$

and if for at least one i the strict inequality holds.

Strictly dominated rows and columns may be deleted.