Optimal control problem

$$\min_u J = \min_u \int_{t_0}^{t_f} L(x(t), u(t))\,dt$$
along $\dot{x} = f(x, u)$ } studied in 332:510

Feb. 5, 99 (3)

Dynamic Games (generalized optimal control)

$$\dot{x} = f(x, u_1, u_2) \quad (1)$$

$$J_1 = \int_{t_0}^{t_f} L_1(x, u_1, u_2)\,dt \quad (2)$$

$$J_2 = \int_{t_0}^{t_f} L_2(x, u_1, u_2)\,dt \quad (3)$$

Two decision makers u_1 and u_2 and two performance criteria, several situations may happen

a) a conflict game with simultaneous decision making (Nash strategies)

$$\min_{u_1} J_1 = \min_{u_1} \int_{t_0}^{t_f} L_1(x, u_1, u_2)\,dt \quad (4)$$

$$\min_{u_2} J_2 = \min_{u_2} \int_{t_0}^{t_f} L_2(x, u_1, u_2)\,dt \quad (5)$$

along trajectories of (1).

b) a conflict game with simultaneous decision making (Stackelberg strategies) with u_1 being the leader and u_2 the follower (or other way around)

c) a cooperative game (Pareto strategy)

$$\min_{\binom{u_1}{u_2}} (\gamma_1 J_1 + \gamma_2 J_2) \quad \text{along trajectories of (1)}$$

d) $J_1 = -J_2 \Longrightarrow$ zero-sum differential game

To study the above problems we need dynamic optimization. The essence of dynamic optimization will be presented in the second part of this course. Source (E. Bryson, "Dynamic Optimization", 1999) - text book for 332:510 to be taught in Spring of 2000.

Chapter 2. FINITE ZERO-SUM GAMES

Section 2.2 Normal Form (Matrix Games)

P1 - player one, matrix rows are his/her alternatives (strategies)
P2 - player two, " columns " " "

$$\overset{m\times n}{A} = \{a_{ij}\}, \quad a_{ij} = \text{game outcomes}$$

Assume that P1 is minimizer
P2 is maximizer

(Ex)

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & -3 & 4 \end{bmatrix}$$

$a_{12} = 1 \Rightarrow$ P1 looses 1 and P2 gains 1
$a_{22} = -3 \Rightarrow$ P1 gains 3 " " looses 3

First we introduce the notion of a <u>security strategy</u>. In that direction, we assume that <u>the game is played only once</u>.

<u>Rational behavior: secure own losses against any strategy of other player.</u>

P1 chooses the row in which the largest element is smaller than the largest element in any other row

(Ex)

$$\begin{array}{cc|c|c|c|c|} & & \multicolumn{4}{c}{P2} \\ & i=1 & 1 & 3 & 3 & -2 \\ P1 & i=2 & 0 & -1 & 2 & 1 \\ & i=3 & -2 & 2 & 0 & 1 \\ & & j=1 & j=2 & j=3 & j=4 \end{array}$$

$\Rightarrow$ $i=2$ or $i=3$ produces a secure strategy for P1

Hence, a secure strategy is not unique, in general.

$$\begin{array}{cc} & P2 \\ P1 & \begin{array}{|c|c|c|c|} \hline 1 & 3 & 3 & -2 \\ \hline 0 & -1 & 2 & 1 \\ \hline -2 & 2 & 0 & 1 \\ \hline \end{array} \end{array}$$

$\Rightarrow$ $j=3$ is a secure strategy for P2

Similarly, P2 chooses the column whose smallest element is no smaller that the smallest element in any other column.

In general of an $A^{m \times n}$ matrix game <u>a secure strategy for P1</u> is

$$\max_j \{a_{i^*j}\} \le \max_j \{a_{ij}\}, \quad i = 1, 2, \dots, m$$

This value of the game represents the <u>upper value of the game</u> and it is denoted by

$$\bar{V}(A) = \max_j \{a_{i^*j}\}$$

<u>A secure strategy for P2 is</u>

$$\min_i \{a_{ij^*}\} \ge \min_i \{a_{ij}\}, \quad j = 1, 2, \dots, n$$

and the corresponding value of the game represents the <u>lower game value</u>

$$\underline{V}(A) = \min_i \{a_{ij^*}\}$$

<u>FACTS</u> (Theorem 2.1)

(i) the security level (upper and lower game values) of each player is unique.

(ii) there exists at least one security strategy for each player.

(iii) $\underline{V}(A) \le \bar{V}(A)$

If P1 plays first, his rational strategy is his security strategy. Hence the game outcome is

P1–P2 $\Rightarrow$ $\bar{V} = \min_{i} \max_{j} \{a_{ij}\}$ = the upper game value

If P2 has to play first, he also applies his security strategy, so that the game outcome in this case is

P2–P1 $\Rightarrow$ $\underline{V} = \max_{j} \min_{i} \{a_{ij}\}$ = the lower game value

In this case where the order of play is established secure strategies lead to an "equilibrium" (the players do not regret their choice of strategies)

What about if the players have to make their decisions simultaneously (independently)?

(P1, P2) $\Rightarrow$?

(Ex)

		P2		
P1		4	0	−1
		0	−1	3
		1	2	1

$i=3$ is the secure strategy for P1 with $\bar{V} = 2$

$j=1$ is the secure strategy for P2 with $\underline{V} = 0$

simultaneous application of $i=3$ and $j=1$ implies

$(i=3, j=1) \Longrightarrow V = 1$

Now, both players might try to achieve the better game value. For example, if P2 sticks with $j=1$, then P1 may try $i=2$ and get $V=0$. Using the same logic, P2 assumes that P1 is going to play his secure strategy and P2 plays $j=2$ which brings him $V=2$. Hence, by taking chances both players hope to improve their performances. This game apparently has no equilibrium and the players keep changing their strategies.

However, if $\underline{V} = \bar{V}$ then despite the simultaneous decision making the game has the equilibrium strategies that are their secure strategies. Such an equilibrium is called the saddle-point

Since the strategies used so far are pure deterministic strategies, the corresponding saddle-point is called the saddle-point in pure strategies

Def. 2.1 Saddle-point in pure strategies satisfy

$$a_{ij^*} \leq a_{i^*j^*} \leq a_{ij^*}$$

with

$$V(A) = a_{i^*j^*}$$

FACTS: (Theorem 2.2)

If $\underline{V}(A) = \bar{V}(A)$ then

(a) ~~Game~~ has a saddle point in pure strategies

(b) $V(A) = \underline{V}(A) = \bar{V}(A)$

Conclusion:

If $\underline{V}(A) \neq \bar{V}(A)$ no equilibrium exists in pure strategies, but it exists in mixed (stochastic) strategies, to be defined now.

MIXED STRATEGIES

If the same game is played over and over again we can assign to each strategy certain probability. For example, P1 plays i_1 with probability y_1, i_2 with probability y_2 and so on. Hence, the strategies become random discrete (finite game) variables with assigned probability distribution. Such strategies are called mixed strategies.

$$u = \begin{cases} i_1 & \text{with probability } y_1 \\ i_2 & \quad '' \qquad\qquad y_2 \\ \vdots & \\ i_m & \quad '' \qquad\qquad y_m \end{cases} \qquad \text{with } \sum_{i=1}^{m} y_i = 1, \quad y_i \geq 0, \forall i$$

mixed strategy of P1.

$$v = \begin{cases} j_1 & \text{with probability } z_1 \\ j_2 & \quad '' \qquad\qquad z_2 \\ \vdots & \\ j_n & \quad '' \qquad\qquad z_n \end{cases} \qquad \text{and } \sum_{i=1}^{n} z_i = 1, \quad z_i \geq 0, \forall i$$

mixed strategy of P2.

The average value of the game outcome is

$$J(y,z) = \sum_{i=1}^{m} \sum_{j=1}^{n} y_i a_{ij} z_j = y^T A z$$

where

$$y = (y_1, \ldots, y_n)^T$$

$$z = (z_1, \ldots, z_n)^T$$

Goals:

$$\left.\begin{array}{ll} P1 & \min_y J(y,z), \quad y_i \geq 0, \quad \sum_{i=1}^{m} y_i = 1 \\ P2 & \max_z J(y,z), \quad z_i \geq 0, \quad \sum_{i=1}^{n} z_i = 1 \end{array}\right\} \text{this is a linear programming problem}$$

$\boxed{\text{Definition 2.3}}$ A vector y^* is called a mixed security strategy for P1 if the following holds

$$\max_{z \in Z} y^{*T} A z \le \max_{Z} y^T A z$$

The upper game value in mixed strategies is

$$\bar{V}_m(A) = \max_{z \in Z} y^{*T} A z \,, \quad Z = \{ z_i : z_i \ge 0, \, \sum z_i = 1$$

This value is called the average security level of P1

Similarly, a vector z^* is a mixed security strategy for P2 with

$$\min_{y \in Y} y^T A z^* \ge \min_{y \in Y} y^T A z \,,$$

and

$$\underline{V}_m(A) = \min_{Y} y^T A z^* \,, \quad Y = \{ y_i ; \, y_i \ge 0, \, \sum y_i = 1 \}$$

is the average security level of P2.

$\boxed{\text{Def 2.4}}$ A saddle-point in mixed strategies is defined by

$$y^{*T} A z \le y^{*T} A z^* \le y^T A z^*$$

with

$$V_m(A) = y^{*T} A z^*$$

being the value of the game in ~~saddle-point~~ mixed strategies

FACTS (Theorem 2.3)

(a) The average security level of each player is unique

(b) There are exists at least one mixed strategy for each player (min or max of a continuous function over a compact set)

(c) $\underline{V}(A) \le \underline{V}_m(A) \le \bar{V}_m(A) \le \bar{V}(A)$