In the case of several equilibria additional criteria have to be imposed in order to make a decision which one to choose as the solution to the game.

b) STACKELBERG STRATEGIES (sequential decision making)

Let P1 be the leader and P2 the follower

The followers problem is very simple

$$\min_{u_2} J_2(\tilde{u}_1, u_2)$$

with given $\tilde{u}_1$.

The leader has a tougher problem. The leader must also solve the followers problem and find the corresponding reaction curve

(a) $\min_{u_2} J_2(u_1, u_2) \Rightarrow u_2 = u_2^r(u_1) \Leftarrow \frac{\partial J_2(u_1,u_2)}{\partial u_2} = 0$

(b) Now the leader has to solve his own optimization problem

$$\min_{u_1} J_1(u_1, u_2^r(u_1)) = \min_{u_1} F_1(u_1)$$

In general the leader assumes that the strategy of the follower will be from the follower's reaction curve and then solves the constrained optimization problem of the former.

The leader's optimization problem is

$$I(u_1, u_2, \lambda) = J_1(u_1, u_2) + \lambda \frac{\partial J_2(u_1, u_2)}{\partial u_2}$$

which leads to

$$\left.\begin{array}{ll} (1) & \dfrac{\partial I}{\partial u_1} = 0 \\ (2) & \dfrac{\partial I}{\partial u_2} = 0 \\ (3) & \dfrac{\partial I}{\partial \lambda} = 0 \end{array}\right\} \Longrightarrow \begin{array}{c} \text{hopefully} \\ (u_1^S, u_2^S, \lambda) \end{array}$$

(1)-(3) can have the unique solution, no solution or many solutions.

The leader's optimization problem can be viewed as <u>constrained optimization problem</u>, which can be solved by using techniques from nonlinear programming.

$$\boxed{\min J_i \quad \text{along the reaction curve } u_j^r(u_i)}$$

Note that the Stackelberg solution is preferred by the leader (the leader can not be worse than for Nash). In general the leader is doing better with the Stackelberg strategy than with the Nash strategy. The follower can be better or worse. If the follower is worse, he would like to play Nash, but the Stackelberg strategy is imposed on him by the rules of the game.

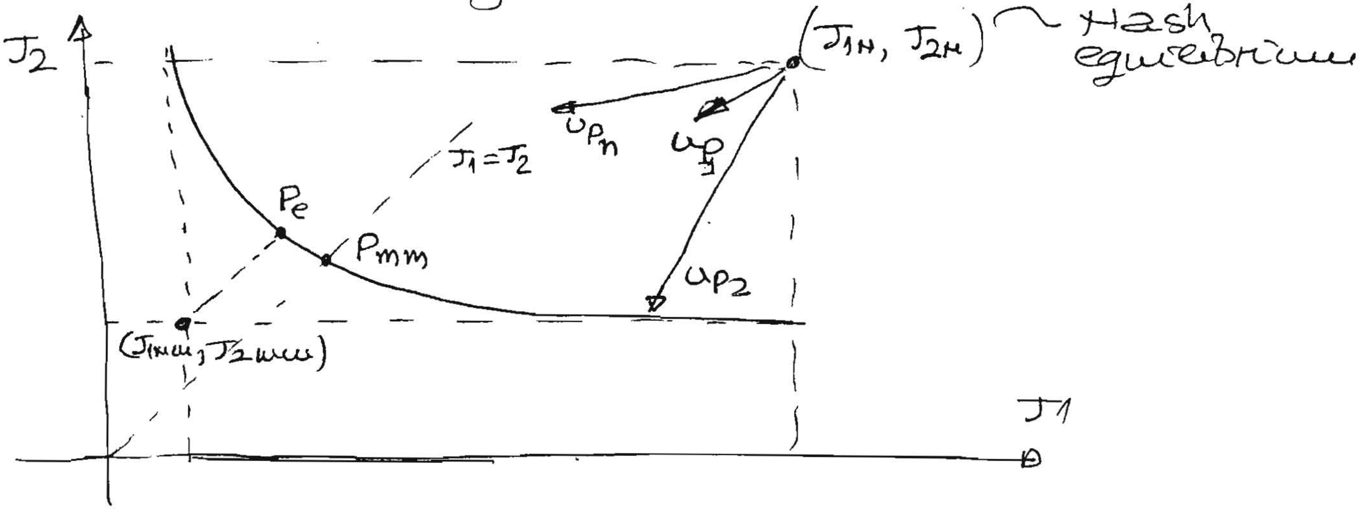
In "Nash figure" since $u_2^r(u_1)$ is (almost) tangent to the iso-cost of the player one, he can not achieve any benefits by taking the role of the leader since $S_1 = N$. However, if P2 is the leader then the iso-cost curve that is tanget to $u_1^r(u_2)$ is inside of his iso-cost Nash curve hence his losses are reduced. Even more, the losses of P1 are also reduced since $J_{S_2} < J_N$.

In some cases both players prefer to be followers (these are so-called stalemate Stackelberg strategies). Hence, in such case none of the players is interesting in announcing his strategy first and they react.

3) COOPERATIVE GAMES
Pareto strategies

It can be seen from the "Nash figure" that the players can do much better than Nash (lightly shaded area). This can happen if they play Stackelberg or if they decide to cooperate. If they decide to cooperate the game solution can be along the dash-dotted line that connects J_{1min} and J_{2min} (absolute minima of J_1 and J_2).

Pareto Strategies (Cooperative Game)



It can be seen from the Nash strategy diagram that both players can minimize losses if they decide to cooperate and optimize

$$J = \gamma_1 J_1 + \gamma_2 J_2, \quad \gamma_1 + \gamma_2 = 1, \ \gamma_1, \gamma_2 > 0$$

Pareto strategy is any strategy that minimizes losses of both (all) players.

$$J_i(u_p) \le J_i(u) \qquad \forall i = 1, 2, \ldots$$

There are infinitely many Pareto strategies (any one that brings the system on or above the $J_2 = f(J_1)$ curve and below $J_1 \le J_{1N}$ and $J_2 \le J_{2N}$. Once we reach the curve $J_2 = f(J_1)$ we can not move any more without increasing one of the J's. At that point we are done. Which point to choose on the $J_2 = f(J_1)$ curve? This leads to different strategies like P_{mm} (minmax Pareto) and P_e (equalization Pareto)