More precisely, there exists the following theorem.

Theorem: let $u_1 \in U_1$ and $u_2 \in U_2$ with $J(u_1, u_2)$ being a continuous function convex with respect to $u_1 \in U_1$ and concave with respect to $u_2 \in U_2$ with U_1 and U_2 being compact (closed and bounded) sets. Then, it exists a point (u_1^*, u_2^*) such that

$$J(u_1^*, u_2) \le J(u_1^*, u_2^*) \le J(u_1, u_2^*)$$

Moreover,

$$J(u_1^*, u_2^*) = \min_{u_1 \in U_1} \max_{u_2 \in U_2} J(u_1, u_2) = \max_{u_1 \in U_1} \min_{u_2 \in U_2} J(u_1, u_2)$$

Recall the notions of convex and concave functions

Def: Convex function $F(y)$

$$F(\lambda y' + (1-\lambda) y'') \le \lambda F(y') + (1-\lambda) F(y'')$$

Concave function $F(y)$

$$F(\mu y' + (1-\lambda) y'') \le \mu F(y') + (1-\mu) F(y'')$$

Note that the optimal game strategies that lead to the saddle point solution are not necessarily unique, but the value of the game is unique.

Let (u_1', u_2') and (u_1'', u_2'') be two equilibrium strategies then

$$J_1(u_1', u_2') = J_1(u_1', u_2'') = J_1(u_1'', u_2') = J_1(u_1'', u_2'')$$

Saddle point necessary conditions

$$\frac{\partial J}{\partial u_1}(u_1, u_2) = 0 \quad \& \quad \frac{\partial J}{\partial u_2}(u_1, u_2) = 0 \Rightarrow \text{saddle point}$$

b) Min max and max min solutions

$$J_1(u_1, u_2) = J(u_1, u_2) \qquad \begin{array}{l} u_1 - \text{minimizer} \\ u_2 - \text{maximizer} \end{array}$$

Here, we have sequential decision making. Let the player I plays first. The player 1 knows that for any $\tilde{u}_1$ the player 2 is going to maximize $J(\tilde{u}_1, u_2)$, that is

$$\max_{u_2} J(\tilde{u}_1, u_2) \implies u_2 = u_2^r(\tilde{u}_1)$$

reaction of player 2 to a strategy of player 1

The function

$$\max_{u_2} J(\tilde{u}_1, u_2) = \cancel{F(\tilde{u}_1)}$$ is called the max function

Now, the player 1 solves his/her problem

$$\min_{u_1} F(u_1) \implies u_1^* = \text{optimal minmax strategy.}$$

Hence, the player 1 solves the problem completely using the following steps:

- find the max function $\cancel{F(u_1)} = \max_{u_2} J(u_1, u_2)$
- find the reaction curve $u_2 = u_2^r(u_1)$
- minimize $J(u_1, u_2^r(u_1)) = F(u_1) \implies u_1^*$

$$\implies u_2^* = u_2^r(u_1^*)$$

The value of the game

$$\min_{u_1} \max_{u_2} J(u_1, u_2) = J(u_1^*, u_2^r(u_1^*)) = \bar{J}^*$$

Similarly, for the maxmin strategy the second player does everything

- Finds $\min_{u_1} J(u_1, \tilde{u}_2)$ = min function
- Finds the reaction curve $u_1 = u_1^r(u_2)$
- ~~minimizes~~ maximize $J(u_1^r(u_2), u_2) = G(u_2)$

$$\underset{u_2}{\cancel{\min}\ \max}\ G(u_2) \Rightarrow u_2^* \Rightarrow u_1^* = u_1^r(u_2^*)$$

The value of the maxmin game is

$$\max_{u_2} \min_{u_1} J(u_1, u_2) = J(u_1^r(u_2^*), u_2^*) = \underline{J}^*$$

Note that in general

$$\max\min(\ ) \leq \min\max(\ )$$

In our case we have

$$\underline{J}^* \leq \bar{J}^*$$

In the case when $\underline{J}^* = \bar{J}^*$ we have the saddle point solution (strategy)

2) CONFLICT GAMES

a) NASH STRATEGIES (simultaneous decision making)

Consider the case of two players $(N=2)$ with the performance criteria

$$J_1(u_1, u_2)$$
$$J_2(u_1, u_2)$$

Both players intend to minimize their loss function

Rational behaviour is that each player minimizes his own losses assuming that the other player does the same, which leads to

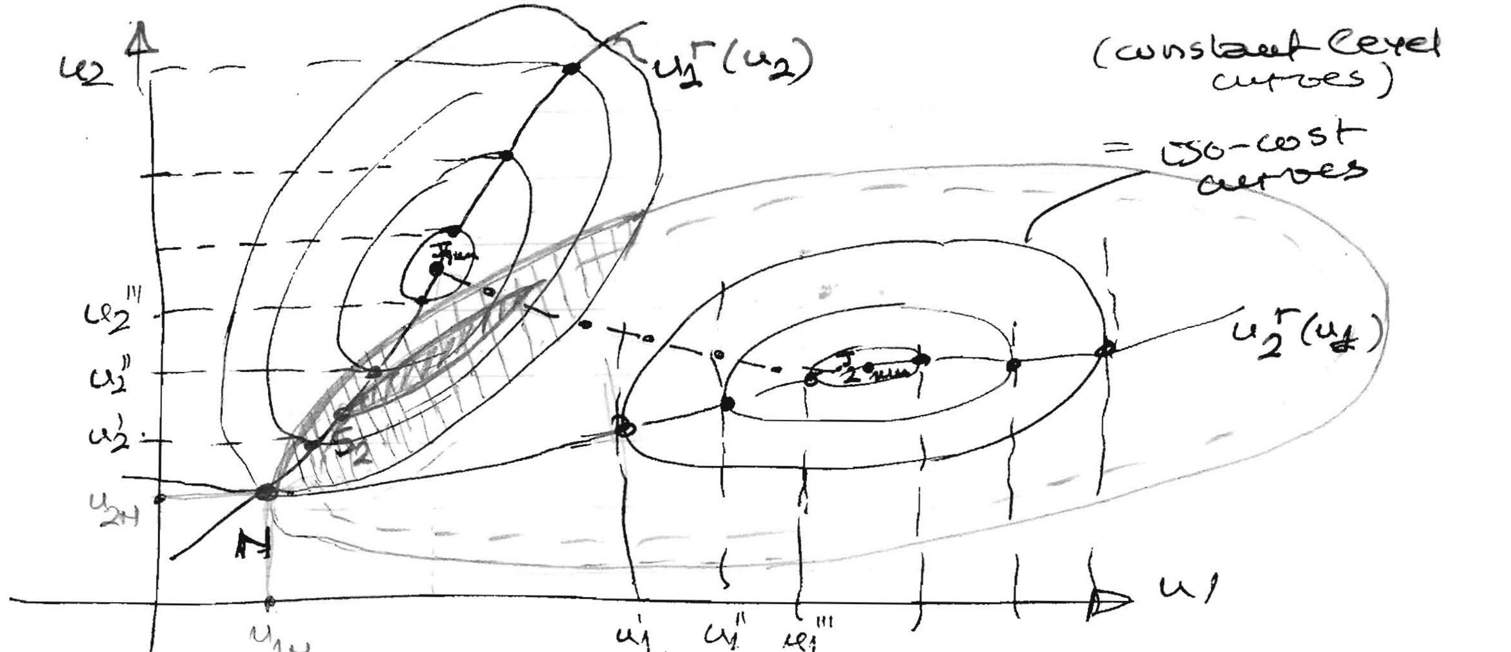
$$J_1(u_1^*, u_2^*) \le J_1(u_1, u_2^*)$$
$$J_2(u_1^*, u_2^*) \le J_2(u_1^*, u_2) \qquad (1)$$

This implies the equilibrium strategies assuming that they exist. To minimize losses each player does the following

$$\left.\begin{aligned} \frac{\partial J_1}{\partial u_1}(u_1, u_2) &= 0 = \varphi_1(u_1, u_2) \\ \frac{\partial J_2}{\partial u_2}(u_1, u_2) &= 0 = \varphi_2(u_1, u_2) \end{aligned}\right\} \Rightarrow \begin{array}{c}(u_1^*, u_2^*) \\ \text{hopefully}\end{array}$$

$\varphi_1(u_1, u_2) = 0 \Rightarrow u_1 = u_1^r(u_2)$ – reaction curve of P1

$\varphi_2(u_1, u_2) = 0 \Rightarrow u_2 = u_2^r(u_1)$ – " " " P2



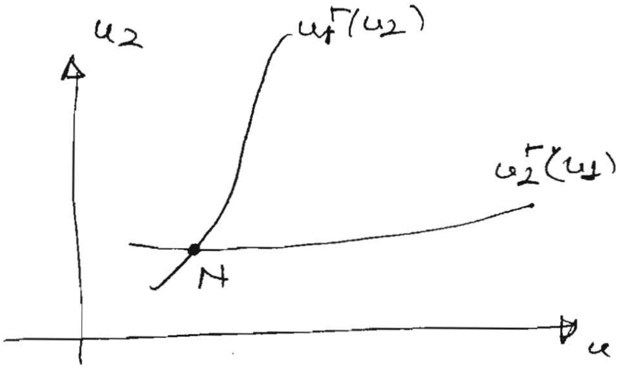
N = Nash equilibrium strategy

S_2 = Stackelber strategy P2-leader and P1-follower

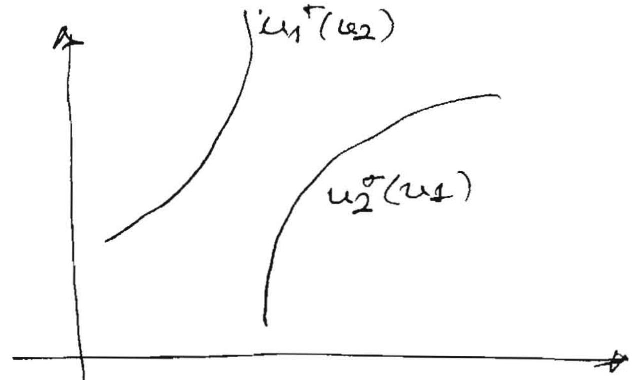
–·–·–·– P = Pareto optimal solutions

This figure is obtained by drawing the constant level curves (iso-cost curves) for $J_1(u_1, u_2)$ and $J_2(u_1, u_2)$ and finding the corresponding reaction curves. For example, for fixed u_1' the minimum for $J_2(u_1', u_2)$ is attained at the point where the line $u_1 = u_1'$ is a tangent to iso-cost curve of P2. Similarly we find ~~the~~ other points on the reaction curve $u_2^r(u_1)$ as well the reaction curve $u_1^r(u_2)$.

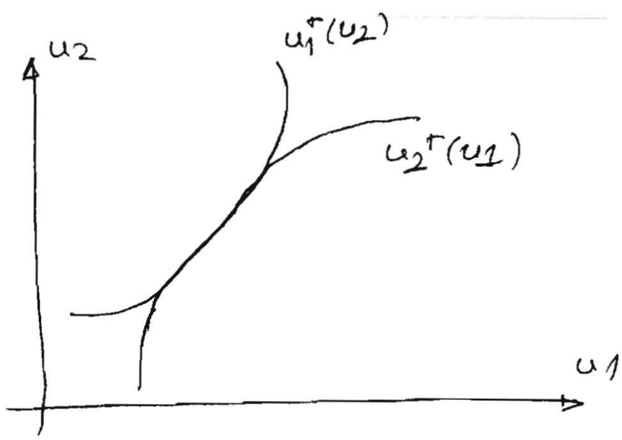
If the intersection of $u_1^r(u_2)$ and $u_2^r(u_1)$ exists and is unique it represents the Nash equilibrium.



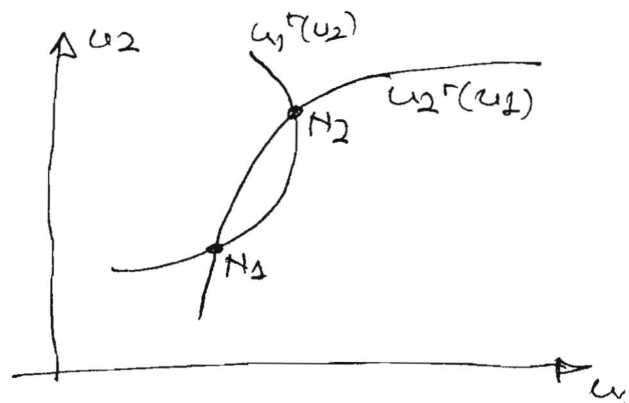
Nash equilibrium



Nash equilibrium does not exist



infinitely many Nash equilibria



two Nash equilibria