Dynamic STACKELBERG Games

Given a dynamic system controlled by two players

$$\dot{x} = f(x, u_1, u_2), \quad x(t_0) = x_0$$

and

$$J_1 = \int_{t_0}^{t_f} L_1(x, u_1, u_2)\,dt + g_1(x(t_f))$$

$$J_2 = \int_{t_0}^{t_f} L_2(x, u_1, u_2)\,dt + g_2(x(t_f))$$

Each player has a performance criterion. Each player minimizes his own criterion. <u>P1 is the leader</u>, and <u>P2 is the follower</u>.

This is the game with sequential decision making. The leader solves the overall optimization problem by taking the optimization problem of the follower as an additional constraint.

<u>FOLLOWER'S OPTIMIZATION PROBLEM</u>

$$H_2 = L_2(x, u_1, u_2) + p_2^T f(x, u_1, u_2)$$

$$(1) \begin{cases} \dot{x} = \dfrac{\partial H}{\partial p_2^T} = f(x, u_1, u_2), & x(t_0) = x_0 \\[2ex] \dot{p}_2^T = -\dfrac{\partial H_2}{\partial x} - \dfrac{\partial H_2}{\partial u_1}\dfrac{\partial u_1}{\partial x}, & p(t_f) = \dfrac{\partial g(x(t_f))}{\partial x} \\[2ex] 0 = \dfrac{\partial H_2}{\partial u_2} \end{cases}$$

these are the necessary conditions for optimum (minimum), as derived in the context of Nash games. We will assume that both players use feedback strategies, hence

$$\frac{\partial u_1}{\partial x} \neq 0 \quad \text{and} \quad \frac{\partial u_2}{\partial x} \neq 0$$

(1) Basically defines the reaction of P2 to any strategy u_1 of P1.

LEADER'S OPTIMIZATION PROBLEM

Static constraint $\frac{\partial H_2}{\partial u} = 0$

$$\tilde{H}_1 = \underbrace{L_1 + p_1^T f}_{= H_1} + \underbrace{\left(\frac{\partial H_2}{\partial x} + \frac{\partial H_2}{\partial u_1}\frac{\partial u_1}{\partial x}\right)\eta}_{\text{comes from a dynamic constraint}} + \frac{\partial H_2}{\partial u}\xi$$

The necessary optimality conditions for the leader are:

$$(2)\begin{cases} \dot{x} = \dfrac{\partial \tilde{H}_1}{\partial p_1^T} = f_1(x, u_1, u_2), \quad x(t_0) = x_0 \\[2ex] \dot{p}_1^T = -\dfrac{\partial \tilde{H}_1}{\partial x} - \dfrac{\partial \tilde{H}_1}{\partial u_1}\dfrac{\partial u_1}{\partial x} - \dfrac{\partial \tilde{H}_1}{\partial u_2}\dfrac{\partial u_2}{\partial x}, \quad p_1(t_f) = \dfrac{\partial g(t_f)}{\partial x} \\[2ex] \dot{p}_2^T = -\dfrac{\partial \tilde{H}_1}{\partial \eta} = -\dfrac{\partial H_2}{\partial x} - \dfrac{\partial H_2}{\partial u_1}\dfrac{\partial u_1}{\partial x}, \quad p_2(t_f) = \dfrac{\partial g(t_f)}{\partial x} \\[2ex] \dot{\eta}^T = -\dfrac{\partial \tilde{H}_1}{\partial p_2} = -\dfrac{\partial H_1}{\partial p_2} - \dfrac{\partial}{\partial p_2}\left[\left(\dfrac{\partial H_2}{\partial x} + \dfrac{\partial H_2}{\partial u_1}\dfrac{\partial u_1}{\partial x}\right)\eta + \dfrac{\partial H_2}{\partial u_2}\xi\right] \end{cases}$$

$$\eta(t_0) = 0$$

$$\left.\begin{aligned} 0 &= \frac{\partial \tilde{H}_1}{\partial u_1} \\ 0 &= \frac{\partial \tilde{H}_1}{\partial u_2} \\ 0 &= \frac{\partial H_2}{\partial u_2} \end{aligned}\right\} \Longrightarrow u_1, u_2, \xi$$

Substituting u_1, u_2, ξ in (2) we get the two point boundary value problem of the form

$$\dot{x} = f \quad , \quad x(t_0) = x_0$$
$$\dot{\eta} = \psi_1 \quad , \quad \eta(t_0) = 0$$
$$\dot{p}_1 = \psi_2 \quad , \quad p_1(t_f) = \frac{\partial g_1}{\partial x}(t_f)$$
$$\dot{p}_2 = \psi_3 \quad , \quad p_2(t_f) = \frac{\partial g_2}{\partial x}(t_f)$$

LINEAR-QUADRATIC STACKELBERG

$$\dot{x} = Ax + B_1 u_1 + B_2 u_2 \, , \quad x(t_0) = x_0$$
$$J_1 = \frac{1}{2}\int_{t_0}^{t_f} (x^T Q_1 x + u_1^T R_{11} u_1 + u_2^T R_{12} u_2)\,dt + \frac{1}{2} x^T(t_f) F_1 x(t_f)$$
$$J_2 = \frac{1}{2}\int_{t_0}^{t_f} (x^T Q_2 x + u_1^T R_{21} u_1 + u_2^T R_{22} u_2)\,dt + \frac{1}{2} x^T(t_f) F_2 x(t_f)$$

$Q_1, Q_2, F_1, F_2 \geq 0$ and symmetric

$R_{11} = R_{11}^T > 0 \quad , \quad R_{22} = R_{22}^T > 0$

$R_{21} = R_{21}^T \geq 0 \quad , \quad R_{12} = R_{12}^T \geq 0$

<u>The followers problem:</u>

$$H_2 = \frac{1}{2}(x^T Q_2 x + u_1^T R_{21} u_1 + u_2^T R_{22} u_2) + p_2^T (Ax + B_1 u_1 + B_2 u_2)$$

follower's necessary conditions

$$\dot{x} = Ax + B_1 u_1 + B_2 u_2 \, , \quad x(t_0) = x_0$$
$$\dot{p}_2 = -\frac{\partial H_2}{\partial x} - \frac{\partial H_2}{\partial u_1}\frac{\partial u_1}{\partial x} = -Q_2 x - A^T p_2 - (B_1^T p_2 + R_{21} u_1)\frac{\partial u_1}{\partial x}$$
$$0 = \frac{\partial H_2}{\partial u_2} = R_{22} u_2 + B_2^T p_2 \implies \boxed{u_2 = -R_{22}^{-1} B_2 p_2}$$

Assume that $\frac{\partial u_1}{\partial x} = 0 \implies$ open-loop strategy for the leader

For $\frac{\partial u_1}{\partial x} \neq 0 \implies$ closed-loop Stackelberg for the leader

The leader's problem:

$$\tilde{H}_1 = \frac{1}{2}\left(x^T Q_1 x + u_1^T R_{11} u_1 + u_2^T R_{12} u_2\right) + p_1^T\left(Ax + B_1 u_1 + B_2 u_2\right)$$

$$+ \underbrace{\left(-\frac{\partial H_2}{\partial x} - \frac{\partial H_2}{\partial u_1}\underbrace{\cancel{\frac{\partial u_1}{\partial x}}}_{0}\right)}_{= -Q_2 x - A^T p_2} \eta^T + \underbrace{\frac{\partial H_2}{\partial u_2}}_{R_{22} u_2 + B_2^T p_2 = 0} \xi$$

$$H_1 = \frac{1}{2}\left(x^T Q_1 x + u_1^T R_{11} u_1 + p_2^T B_2 R_{22}^{-1} R_{12} R_{22}^{-1} B_2^T p_2\right) - \left(Q_2 x + A^T p_2\right)\eta$$

So far we have

$$(3)\begin{cases} \dot{x} = Ax - B_2 R_{22}^{-1} B_2^T p_2 + B_1 u_1 \\ \dot{p}_2 = -Q_2 x - A^T p_2 \end{cases}$$

and

$$(4)\left\{ J_1 = \frac{1}{2}\int_{t_0}^{t_f}\left(x^T Q_1 x + u_1^T R_{11} u_1 + p_2^T B_2 R_{22}^{-1} R_{11} R_{22}^{-1} B_2^T p_2\right) dt + \frac{1}{2} x^T(t_f) F_1 x(t_f) \right.$$

(3) and (4) represent an augmented linear system

$$\begin{bmatrix} \dot{x} \\ \dot{p}_2 \end{bmatrix} = \overbrace{\begin{bmatrix} A & -B_2 R_{22}^{-1} B_2^T \\ -Q_2 & -A^T \end{bmatrix}}^{\mathcal{A}} \begin{bmatrix} x \\ p_2 \end{bmatrix} + \overbrace{\begin{bmatrix} B_1 \\ 0 \end{bmatrix}}^{\mathcal{B}} u_1$$

and an augmented quadratic performance criterion

$$J_1 = \frac{1}{2}\int_{t_0}^{t_f}\left(\begin{bmatrix} x \\ p_2 \end{bmatrix}^T \begin{bmatrix} Q_1 & 0 \\ 0 & \underbrace{B_2 R_{22}^{-1} R_{11} R_{22}^{-1} B_2^T}_{\mathcal{Q}} \end{bmatrix} \begin{bmatrix} x \\ p_2 \end{bmatrix} + u_1^T R_{11} u_1\right) dt + \frac{1}{2}\begin{bmatrix} x(t_f) \\ p_2(t_f) \end{bmatrix}^T \begin{bmatrix} F_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t_f) \\ p_2(t_f) \end{bmatrix}$$

$$\Longrightarrow u_1^{opt} = -R_{11}^{-1}\begin{bmatrix} B_1 \\ 0 \end{bmatrix} K \begin{bmatrix} x(t) \\ p_2(t) \end{bmatrix}$$

from the TPBVP of the state-costate eqs. (open-loop control for P1)

$$\mathcal{A}^T K + K\mathcal{A} + \mathcal{Q} - K\mathcal{B}R_{11}^{-1}\mathcal{B}^T K = 0$$

$2n \times 2n$

note K is doubled dimension