Lemma: $V = \int_0^\infty e^{A^T t} Q e^{At} dt$

then $A^T V + V A + Q = 0$

Proof: Start with

$$\frac{d}{dt}\left(e^{A^T t} Q e^{At}\right) = A^T e^{A^T t} Q e^{At} + e^{A^T t} Q e^{At} A$$

and integrate from 0 to $+\infty$

$$\int_0^{+\infty} d\left(e^{A^T t} Q e^{At}\right) = -Q = A^T \underbrace{\int_0^\infty \left(e^{A^T t} Q e^{At}\right) dt}_{=V} + \underbrace{\int_0^\infty \left(e^{A^T t} Q e^{At}\right) dt}_{=V} A$$

Hence

$$-Q = A^T V + V A$$

ZERO-SUM DIFFERENTIAL GAMES

$$\dot{x} = f(x, u, v) \quad , \; x(t_0) = x_0$$

$$J(x(t_0), t_0) = \int_{t_0}^{t_f} L(x, u, v) dt + g(x(t_f))$$

P1 intends to minimize J, hence his goal is to find u such that

$$\min_u \left\{ \int_{t_0}^{t_f} L(x, u, v) dt + g(x(t_f)) \right\}$$

is achieved. The other player maximizes J, that is

$$\max_v \left\{ \int_{t_0}^{t_f} L(x, u, v) dt + g(x(t_f)) \right\}$$

Let us define the optimal performance value

$$J^*(x(t_0), t_0) = \min_u \max_v \left\{ \int_{t_0}^{t_f} L(x, u, v) dt + g(x(t_f)) \right\}$$

$$\triangleq \int_{t_0}^{t_f} L(x^*, u^*, v^*) dt + g(x^*(t_f))$$

Let v uses his optimal strategy from to t_0 t_f, then

$$J^*(x(t_0),t_0) = \min_u \left\{ \int_{t_0}^{t_0+\Delta t} L(x,u,v^*)dt + \underbrace{\int_{t_0+\Delta t}^{t_f} L(x^*,u^*,v^*)dt + g(x^*(t_f))}_{\triangleq J^*(x(t_0+\Delta t),\, t_0+\Delta t)} \right\}$$

$J^*(x(t_0),t_0)$: any initial state and any initial time

we can expand $J^*(x(t_0+\Delta t), t_0+\Delta t)$ in a Taylor series, which leads to

$$J^*(x(t_0),t_0) = \min_u \left\{ \int_{t_0}^{t_0+\Delta t} L(x,u,v^*)dt + J^*(x(t_0),t_0) + \frac{\partial J^*}{\partial t}\Delta t + \left(\frac{\partial J^*}{\partial x}(x(t_0),t_0)\right)^T \times [x(t+\Delta t) - x(t)] + \text{h.o.t} \right\}$$

$\times$ multiplying

for Δt small we can also approximate the integral by $L(x,u,v^*)\Delta t$, so that

$$\cancel{J^*(x(t_0),t_0)} = \min_u \left\{ L(x,u,v^*)\Delta t + \cancel{J^*(x(t_0),t_0)} + \frac{\partial J^*}{\partial t_0}\Delta t + \frac{\partial J^*}{\partial x}(x(t_0),t_0)\,[x(t_0+\Delta) - x(t)] + \text{h.o.t} \right\}$$

$$-\frac{\partial J^*}{\partial t}\Delta t = \min_u \left\{ L(x,u,v^*)\Delta t + \left(\frac{\partial J^*}{\partial x}(x(t_0),t_0)\right)[x(t_0+\Delta) - x(t)] + \text{h.o.t} \right\}$$

In the limit when $\Delta t \to 0$ we have

$$-\frac{\partial J^*}{\partial t_0} = \min_u \left\{ L(x,u,v^*) + \left(\frac{\partial J^*}{\partial x}(x(t_0),t_0)\right)^T \dot{x}(t_0) \right\} + \cancel{\text{h.o.t}}^{\,0}$$

since t_0 is any initial time we can take $t_0 = t$ and $x(t_0)$ is any initial state so that

$$\boxed{-\frac{\partial J^*}{\partial t} = \min_u \left\{ L(x,u,v^*) + \left(\frac{\partial J^*}{\partial x}(x(t),t)\right)^T f(x(t),u(t),v^*(t)) \right\}}$$

This is the Isaacs equation for zero-sum differential games. Note it was derived at the beginning of the 1950s.

Similarly, for P_2 the Isaacs equation is

$$\boxed{-\frac{\partial J^*}{\partial t} = \min_{v} \left\{ L(x, u^*, v) + \left(\frac{\partial J^*}{\partial x}(x(t),t)\right)^T f(x(t), u^*(t), v(t)) \right\}}$$

Putting these two equations together we have

$$-\frac{\partial J^*}{\partial t} = \min_u \max_v \left\{ L(x,u,v) + \left(\frac{\partial J^*}{\partial x}\right)^T f(x,u,v) \right\}$$

or under the optimal controls u^* and v^* the Isaacs equation has the form

$$-\frac{\partial J^*}{\partial t} = L(x^*, u^*, v^*) + \left(\frac{\partial J^*}{\partial x}\right)^T f(x^*, u^*, v^*)$$

in general hard for solving the same story as befor

Let us take the partial derivatives with respect to x of the above (omitting $*$ for simplicity)

$$-\frac{\partial^2 J}{\partial x \partial t} = \frac{\partial L}{\partial x} + f \frac{\partial^2 J}{\partial x^2} + \left(\frac{\partial J}{\partial x}\right)^T \cdot \frac{\partial f}{\partial x}$$

$$+ \underbrace{\frac{\partial}{\partial u}\left\{L + \frac{\partial J}{\partial x} \cdot f\right\}}_{=0} \frac{\partial u}{\partial x} + \underbrace{\frac{\partial}{\partial v}\left\{L + \frac{\partial J}{\partial x} f\right\}}_{=0} \frac{\partial v}{\partial x}$$

$$-\frac{\partial^2 J}{\partial x \partial t} + f \frac{\partial^2 J}{\partial x^2} = \frac{\partial L}{\partial x} + \frac{\partial J}{\partial x} \cdot \frac{\partial f}{\partial x}$$

Introduce the costate variable $p = \frac{\partial J}{\partial x}^T$

$$-\dot{p}^T = \frac{\partial}{\partial x} L + p^T \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \underbrace{\left(L + p^T f\right)}_{\triangleq H = \text{Hamiltonian}}$$

$$-\dot{p} = -\left(\frac{\partial H}{\partial x}\right)^T$$

Note that $p(t_f) = \frac{\partial g(x(t_f))}{\partial x}$

Summary: In terms of Hamiltonian $H = L + p^T f$ the necessary conditions for optimum (saddle point) are

$$\dot{x} = \frac{\partial H}{\partial p^T} = f(x, u, v), \quad x(t_0) = 0$$

$$\dot{p} = -\left(\frac{\partial H}{\partial x}\right)^T, \quad p(t_f) = g(x(t_f))$$

$$0 = \frac{\partial H}{\partial u}$$

$$0 = \frac{\partial H}{\partial v}$$

LINEAR-QUADRATIC

$$\dot{x} = Ax + B_1 u + B_2 v$$

$$J = \frac{1}{2}\int_0^\infty (x^T Q x + u^T R_1 u - v^T R_2 v)\,dt, \quad R_1 > 0, \ R_2 > 0$$

Form the Hamiltonian

$$H = \frac{1}{2}(x^T Q x + u^T R_1 u - v^T R_2 v) + p^T (Ax + B_1 u + B_2 v)$$

$$\left.\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p} = Ax + B_1 u + B_2 v, \quad && x(t_0) = x_0 \\ \dot{p} &= -\frac{\partial H}{\partial x} = -A^T p - Qx && p(\cancel{t_f}) = 0 \end{aligned}\right\}$$

$$0 = \frac{\partial H}{\partial u} = R_1 u + B_1^T p \implies u = -R_1^{-1} B_1^T p$$

$$0 = \frac{\partial H}{\partial v} = -R_2 v + B_2^T p \implies v = +R_2^{-1} B_2^T p$$

$$\left.\begin{aligned} \dot{x} &= Ax - \underbrace{B_1 R_1^{-1} B_1^T}_{S_1} p + \underbrace{B_2 R_2^{-1} B_2^T}_{S_2} p = Ax + (S_2 - S_1) p \\ \dot{p} &= -Qx - A^T p \end{aligned}\right\}$$

This two point boundary value method can be solved by using the Riccati formalism

$$p(t) = P x(t)$$

$$\dot{p}(t) = P \dot{x}(t)$$

$$-Qx - A^T p = P(Ax + (S_2 - S_1) p)$$

$$-Qx - A^T P x = PAx + P(S_2 - S_1) P x$$

Since this must hold for any x, we have

$$\boxed{0 = A^T P + PA + Q - P(S_2 - S_1) P}$$

which represents the algebraic Riccati equation of zero-sum differential games. (called the generalized algebraic Riccati equation)

Note that $S_2 - S_1$ is undefinite, which makes this equation much more difficult for solving and analysing than the standard algebraic Riccati equation.

Algorithm of (Li and Gajic, 1995; "New Trends in Dynamic Games and Applications, (ed) Olsder, Birkhauser, Boston) finds the solution of the above generalized Riccati equation

$$\underbrace{(A - S_2 P^{(i)})^T P^{(i+1)} + P^{(i+1)} (A - S_2 P^{(i)})}_{\text{Lyapunov equation for } P^{(i+1)}} + (Q + P^{(i)} S_1 P^{(i)} + P^{(i)} S_2 P^{(i)}) =$$

with

$$A^T P^{(0)} + P^{(0)} A + Q - P^{(0)} S_2 P^{(0)} = 0 = \text{standard algebraic Riccati equation.}$$

just for your inf. not for exa...

The algorithm converges for any $(A, \sqrt{S}, \sqrt{Q})$ stabilizable-detectable (controllable-observable), assuming that the positive semidefinite stabilizing solution exist