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FEEDBACK and FEEDBACK/OPEN-LOOP NASH STRATEGIES

We have derived that on the optimal trajectory we have (stars are dropped for simplicity)

$$\dot{x} = Ax - S_1 p_1 - S_2 p_2 \qquad x(t_0) = x_0$$

$$\dot{p}_1 = -Q_1 x - A^T p_1 - \left(\frac{\partial u_2}{\partial x}\right)^T \left(B_2^T p_1 + R_{12} u_2\right) \qquad p_1(t_f) = F_1 x(t_f)$$

$$\dot{p}_2 = -Q_2 x - A^T p_2 - \left(\frac{\partial u_1}{\partial x}\right)^T \left(B_1^T p_2 + R_{21} u_1\right) \qquad p_2(t_f) = F_2 x(t_f)$$

For the open-loop Nash strategies

$$\frac{\partial u_1}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u_2}{\partial x} = 0$$

The feedback strategies $(u_1 = u_1(x),\ u_2(x) = u_2)$ imply

$$\frac{\partial u_1}{\partial x} \neq 0 \quad \text{and} \quad \frac{\partial u_2}{\partial x} \neq 0$$

We have already shown that

$$\frac{\partial H_1}{\partial u_1} = 0 \implies u_1^* = -R_{11}^{-1} B_1^T p_1^*$$

$$\frac{\partial H_2}{\partial u_2} = 0 \implies u_2^* = -R_{22}^{-1} B_2^T p_2^*$$

Since the way to solve the TPBVP ~~problem~~ problem is to use

$$p_1 = K_1 x \quad \text{and} \quad p_2 = K_2 x$$

we get at the same price the feedback (optimal) strategies

$$u_1^* = -R_{11}^{-1} B_1^T K_1 x^*$$

$$u_2^* = -R_{22}^{-1} B_2^T K_2 x^*$$

This implies

$$\frac{\partial u_1^*}{\partial x} = -R_{11}^{-1} B_1^T K_1 \qquad \frac{\partial u^*}{\partial x} = -R_{22}^{-1} B_2^T K_2$$

Now the state-costate optimal equations corresponding to the feedback strategies become

$$\dot{x} = Ax - S_1 p_1 - S_2 p_2$$

$$\dot{p}_1 = -Q_1 x - A^T p_1 + K_2 \underbrace{B_2 R_{22}^{-1} B_2^T}_{S_2} p_1 - K_2 \underbrace{B_2 R_{22}^{-1} R_{12} \cdot R_{22}^{-1} B_2^T}_{Z_2} K_2 x$$

$$\dot{p}_2 = -Q_2 x - A^T p_2 + K_1 \underbrace{B_1 R_{11}^{-1} B_1^T}_{S_1} p_2 - K_1 \underbrace{B_1 R_{11}^{-1} R_{21} R_{11}^{-1} B_1^T}_{Z_1} K_1 x$$

$$\dot{x} = Ax - S_1 p_1 - S_2 p_2 \qquad x(t_0) = 0$$

$$\dot{p}_1 = -Q_1 x - K_2 Z_2 K_2 x - A^T p_1 + K_2 S_2 p_1 \qquad p_1(t_f) = F_1 x(t_f)$$

$$\dot{p}_2 = -Q_2 x - K_1 Z_1 K_1 x - A^T p_2 + K_1 S_1 p_2 \qquad p_2(t_f) = F_2 x(t_f)$$

or

$$\begin{bmatrix} \dot{x} \\ \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} A - S_1 K_1 - S_2 K_2 & 0 & 0 \\ -(Q_1 + K_2 Z_2 K_2) & -(A - S_2 K_2)^T & 0 \\ -(Q_2 + K_1 Z_1 K_1) & 0 & (A - S_1 K_1)^T \end{bmatrix} \begin{bmatrix} x \\ p_1 \\ p_2 \end{bmatrix}, \qquad \begin{array}{l} x(t_0) = x_0 \\ p_1(t_f) = F_1 x(t_f) \\ p_2(t_f) = F_2 x(t_f) \end{array}$$

This two-point-boundary value problem can be solved by using

$$p_1 = K_1 x \quad \text{and} \quad p_2 = K_2 x$$

or

$$\dot{p}_1 = \dot{K}_1 x + K_1 \dot{x} \quad \text{and} \quad \dot{p}_2 = \dot{K}_2 x + K_2 \dot{x}$$

which leads to

$$\dot{p}_1 = \dot{K}_1 x + K_1 \dot{x}$$

$$-(Q_1 + K_2 Z_2 K_2)x \; - (A - S_2 K_2)^T \underbrace{p_1}_{= K_1 x} = \dot{K}_1 x + K_1 (A - S_1 K_1 - S_2 K_2) x$$

since this must hold for any x we have

$$-Q_1 - K_2 Z_2 K_2 - A^T K_1 + K_2 S_2 K_1 = \dot{K}_1 + K_1 A - K_1 S_1 K_1 - K_1 S_2 K_2$$

or

$$\boxed{-\dot{K}_1 = K_1 A + A^T K_1 + Q_1 - K_1 S_1 K_1 - K_2 S_2 K_1 - K_1 S_2 K_2 + K_2 Z_2 K_2 \qquad K_1(t_f) = F_1}$$

also $p_1(t_f) = K_1 x(t_f) = F_1 x(t_f) \Rightarrow$

Similarly, for P_2 we have

$$\boxed{-\dot{K}_2 = K_2 A + A^T K_2 + Q_2 - K_2 S_2 K_2 - K_1 S_1 K_2 - K_2 S_1 K_1 + K_1 Z_1 K_1, \quad K_2(t_f) = F_2}$$

As, before these coupled differential equations can be solved numerically rather easily integrating backward in time.

If $t_f \to \infty$ we get $(\dot{K}_1 = 0, \dot{K}_2 = 0)$ the system of algebraic coupled Riccati equations

$$\left.\begin{aligned} 0 &= K_1 A + A^T K_1 + Q_1 - K_1 S_1 K_1 - K_2 S_2 K_1 - K_1 S_2 K_2 + K_2 Z_2 K_2 \\ 0 &= K_2 A + A^T K_2 + Q_2 - K_2 S_2 K_2 - K_1 S_1 K_2 - K_2 S_1 K_1 + K_1 Z_1 K_1 \end{aligned}\right\}$$

These equations are still the subject of research. An efficient algorithm for their numerical solution is presented by (Li and Gajic, 1988). The algorithm is given in terms of decoupled algebraic Lyapunov equations (note that here $K_1 = K_1^T$, $K_2 = K_2^T$, which was not the case for the open-loop strategies)

$$(A - S_1 K_1^{(i)} - S_2 K_2^{(i)})^T K_1^{(i+1)} + K_1^{(i+1)} (A - S_1 K_1^{(i)} - S_2 K_2^{(i)})$$
$$= -(Q_1 + K_1^{(i)} S_1 K_1^{(i)} + K_2^{(i)} Z_2 K_2^{(i)})$$

$$(A - S_1 K_1^{(i)} - S_2 K_2^{(i)})^T K_2^{(i+1)} + K_2^{(i+1)} (A - S_1 K_1^{(i)} - S_2 K_2^{(i)})$$
$$= -(Q_2 + K_1^{(i)} Z_1 K_1^{(i)} + K_2^{(i)} S_2 K_2^{(i)})$$

with the initial conditions obtained from

$$K_1^{(0)} A + A^T K_1^{(0)} + Q_1 - K_1^{(0)} S_1 K_1^{(0)} = 0$$
$$K_2^{(0)} (A - S_1 K_1^{(0)}) + (A - S_1 K_1^{(0)})^T K_2^{(0)} + (Q_2 + K_1^{(0)} Z_1 K_1^{(0)}) - K_2^{(0)} S_2 K_2^{(0)} = 0$$

This can be easily programmed in MATLAB since MATLAB has built in functions lyap and are for solving algebraic Lyapunov and Riccati equations, for example

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K10 = are (A, S1, Q1)
K1I = lyap((A-S1*K10-S2*K20)', Q1+K10*S1*K10+K20*Z2*K20)
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More details about this algorithm can be found in "New Trends in Dynamic Games and Applications" pp. 333-351, G. Olsder, editor, Birkhauser, Boston, 1995.

In summary, the closed loop feedback strategies are given by

$$u_1^* = -R_{11}^{-1} B_1^T K_1 x$$
$$u_2^* = -R_{22}^{-1} B_2^T K_2 x$$

$$\dot{x} = (A - S_1 K_1 - S_2 K_2) x, \quad x(t_0) = x_0$$

where K_1 and K_2 are <u>stabilizing</u> ($A - S_1K_1 - S_2K_2$ must be stable) solutions of the coupled algebraic Riccati equations.

For the feedback/ open-loop Nash strategies one of the players uses the feedback strategy and another one uses the open-loop strategy. Let $u_1 = u_1(x)$ and $u_2 = u_2(t)$ then

$$\frac{\partial u_1}{\partial x} \neq 0 \quad \text{and} \quad \frac{\partial u_2}{\partial x} = 0$$

$$\Rightarrow \begin{cases} \dot{x} = Ax - S_1 p_1 - S_2 p_2, \quad x(t_0) = x_0 \\ \dot{p}_1 = -Q_1 x - A^T p_1 \\ \dot{p}_2 = -Q_2 x - A^T p_2 - K_1 B_1 R_{11}^{-1} \left(B_1^T p_2 - R_{21} R_{11}^{-1} B_1^T K_1 x \right) \end{cases}$$

$$\Rightarrow -\dot{K}_2 = K_2 A + A^T K_2 + Q_2 - K_2 S_2 K_2 - K_2 S_1 K_1 - K_1 S_1 K_2 + K_1 Z_1 K_1 \qquad K_2(t_f) = F_2$$

$$-\dot{K}_1 = K_1 A + A^T K_1 + Q_1 - K_1 S_1 K_1 - K_1 S_2 K_2, \qquad K_1(t_f) = F_1$$

The symmetry is lost, that is $K_1 \neq K_1^T \Rightarrow K_2 \neq K_2^T$

Solving the corresponding algebraic coupled Riccati equations is difficult.

$$\left.\begin{aligned} 0 &= K_2 A + A^T K_2 + Q_2 - K_2 S_2 K_2 - K_2 S_1 K_1 - K_1 S_1 K_2 + K_1 Z_1 K_1 \\ 0 &= K_1 A + A^T K_1 + Q_1 - K_1 S_1 K_1 - K_1 S_2 K_2 \end{aligned}\right\} \Rightarrow K_1, K_2$$

The optimal strategies are

$$u_1^* = -R_{11} B_1^T K_1 x^*(t)$$

$$u_2^* = -R_{11} B_1^T p_2^*(t)$$

where x^* and p_2^* satisfy the above defined TPBVP.

HOW TO FIND J_1^* and J_2^* ?

$$J_1^*(x(t_0),t_0) = \frac{1}{2}\int_{t_0}^{t_f} (x^{T*} Q_1 x^* + u_1^{T*} R_{11} u_1^* + u_2^{T*} R_{12} u_2^*)\,dt + \frac{1}{2} x^T(t_f) F_1 x^*(t_f)$$

$$\dot{x}^*(t) = (A - S_1 K_1 - S_2 K_2) x^*, \quad x^*(t_0) = x_0$$

take for simplicity $t_0 = 0$, $t_f = \infty$, $x(t_f) = 0$

$$u_1^* = -R_{11}^{-1} B_1^T K_1 x^*$$
$$u_2^* = -R_{22}^{-1} B_2^T K_2 x^*$$

$\Longrightarrow$

$$J_1^*(x(0),0) = \frac{1}{2}\int_0^\infty x^{*T} (Q_1 + K_1 S_1 K_1 + K_2 S_2 K_2) x^* \,dt$$

From the state equation we have

$$x^*(t) = e^{(A - S_1 K_1 - S_2 K_2) t} x(0)$$

$\Longrightarrow$

$$J_1^*(x(0),0) = \frac{1}{2} x^T(0) \underbrace{\int_0^\infty e^{(A - S_1 K_1 - S_2 K_2)^T t} (Q_1 + K_1 S_1 K_1 + K_2 S_2 K_2) e^{(A - S_1 K_1 - S_2 K_2) t}\,dt}_{?}$$

Hence

$$J_1^*(x(0),0) = \frac{1}{2} x^T(0) \,(\,?\,)\, x(0)$$

It can be shown that (?) satisfies the algebraic Lyapunov equation

$$(A - S_1 K_1 - S_2 K_2)^T (?) + (?)(A - S_1 K_1 - S_2 K_2) + Q_1 + K_1 S_1 K_1 + K_2 S_2 K_2$$

since the original algebraic Nash game Riccati can be written as

$$(A - S_1 K_1 - S_2 K_2)^T K_1 + K_1 (A - S_1 K_1 - S_2 K_2) + Q_2 + K_2 S_2 K_2 + K_1 Z_1 K_1$$

we conclude $(?) = K_1$ similary

$$J_1^* = \frac{1}{2} x^T(0) K_1 x(0) \qquad J_2^* = \frac{1}{2} x^T(0) K_2 x(0)$$