Feb. 27, 97 (6)

(4.3) KALMAN-BUCY FILTER

from the original paper

(4.4) OPTIMAL LINEAR PREDICTORS

$$\hat{x}_k(+) = \Phi_{k-1}\hat{x}_{k-1}(+) + \bar{K}_k(z_k - H_k\hat{x}_k(-)) \equiv \text{Kalman-filter}$$

not available (equivalent to $\bar{K}_k = 0$)

$\Rightarrow$ predictor equation

$$\boxed{\hat{x}_k(+) = \Phi_{k-1}\hat{x}_{k-1}(+)}$$

In the continuous-time domain

$$\dot{\hat{x}}(t) = F(t)\hat{x}(t) + \bar{K}(t)(z(t) - H\hat{x}(t))$$

not available ($\bar{K} = 0$)

$\Rightarrow$ continuous-time predictor

$$\boxed{\dot{\hat{x}}(t) = F(t)\hat{x}(t)}$$

previously estimated values represent the initial conditions for predictors.

(4.5.1) CORRELATION BETWEEN PLANT AND MEASUREMENT NOISE

Problem formulation:

$$\left\{\begin{array}{l} x_k = \Phi_{k-1}x_{k-1} + w_{k-1} \\ z_k = H_k x_k + v_k \\ E\{w_k v_i^T\} = C_k \Delta(k-i) \end{array}\right\}$$

+ standard assumptions previously used for the Kalman filter

$E\{x_0\} = x_0$

$Var(x_0) = P_0$

$E\{w_k\} = 0, \quad E\{v_k\} = 0$

$E\{w_k w_i^T\} = Q_k \Delta(k-i)$

$E\{v_k v_i^T\} = R_k \Delta(k-i)$

Solution: Using the orthogonality principle, like for the standard Kalman filter we get a little bit more complex results.

Results:

$$\hat{x}_k(+) = \hat{x}_k(-) + \bar{K}_k\,(z_k - H_k\hat{x}_k(-)) \quad = \text{the same as before}$$

$$\bar{K}_k = (P_k(-)H_k^T + \underbrace{C_k}_{\text{new}})\,(H_k P_k(-) H_k + R_k + \underbrace{H_k C_k}_{\text{new}} + \underbrace{C_k^T H^T}_{\text{new}})^{-1}$$

$$P_k(+) = P_k(-) - \bar{K}_k\,(H_k P_k(-) + \underbrace{C_k^T}_{\text{new}})$$

Continuous-time Domain

$$\dot{x} = F(t)x + G(t)w(t)$$

$$z = H(t)x + v(t)$$

$$E\{w(t)v^T(\tau)\} = C\,\delta(t-\tau)$$

$$\Longrightarrow \dot{\hat{x}}(t) = F(t)\hat{x}(t) + \bar{K}(t)\,(z(t) - H\hat{x}(t)) \qquad \text{the same as before}$$

$$\bar{K}(t) = (P(t)H^T(t) + \underbrace{C(t)}_{\text{new term}})\,R^{-1}(t)$$

$$\dot{P}(t) = (F - \overbrace{C}^{\text{new}}R^{-1}H)P(t) + P(t)(F - \overbrace{C}^{\text{new}}R^{-1}H) - P(t)H^T R^{-1} H P(t) + G(Q - \underbrace{C R^{-1} C^T}_{\text{new}})G^T$$

more complex Riccati equation and slightly different filter gain

4.5.2 Colored Noise in MEASUREMENTS

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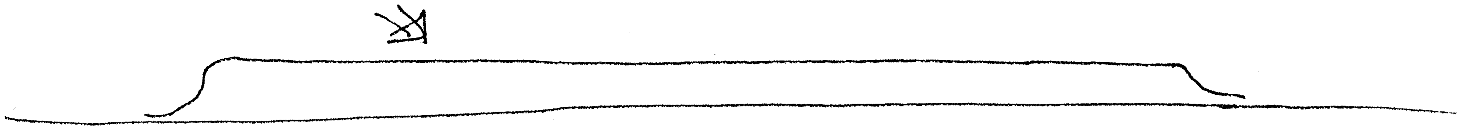
wide band noise (also called exponentially correlated noise)



$$x_k = \Phi_{k-1} x_{k-1} + w_{k-1}$$

$$z_k = H_k x_k + \underbrace{v_k}_{\text{colored noise}}$$

Use a shaping filter to model the colored noise as an output of a system driven by white noise that is

$$v_k = A_{k-1} v_{k-1} + \underbrace{\eta_{k-1}}_{\text{zero-mean Gaussian white noise}}$$

Using the augmented system, we get

$$\begin{bmatrix} x_k \\ v_k \end{bmatrix} = \begin{bmatrix} \Phi_{k-1} & 0 \\ 0 & A_{k-1} \end{bmatrix} \begin{bmatrix} x_{k-1} \\ v_{k-1} \end{bmatrix} + \begin{bmatrix} w_{k-1} \\ \eta_{k-1} \end{bmatrix}$$

$$z_c = [H_k \; \vdots \; I] \begin{bmatrix} x_k \\ v_k \end{bmatrix}$$ note no measurement noise

$$\left.\begin{array}{l} (\mathcal{X}_k) = \mathcal{A}_{k-1} \mathcal{X}_{k-1} + \mathcal{W}_{k-1} \\ z_c = [\quad] \mathcal{X}_k \end{array}\right\}$$ formulation of the filtering problem in the original paper by Kalman

All results are the same assuming that

$$[H_k P_k(-) H_k^T + 0]^{-1}$$ exists

replaced by $R > 0$ in noisy measurement case

That is, we need

$H_k P_k(-) H_k^T$ invertible for $\forall k$

NOTE THAT THE CONTINUOUS-TIME EQUIVALENT

$$\left.\begin{array}{l} \dot{x} = Fx + Gw \\ z = Hx \end{array}\right\}$$ HAS NO SOLUTION

can be studied as a limited case of $z = Hx + \epsilon$, $\epsilon \to 0$

~~4.7~~ QUADRATIC LOSS FUNCTION.

4.8 CONTINUOUS-TIME RICCATI EQUATION

"The need to solve the Riccati equation is perhaps the greatest single cause of anxiety and agony on the part of people faced with implementation of the Kalman filter" — TEXT BOOK

4.8.1 DIFFERENTIAL RICCATI EQUATION

$$\dot{P} = PF^T + FP + Q - PH^T R^{-1} H P \quad , \quad P(t_0) = P_0$$

In general all matrices are functions of time.

We look for the solution of the above nonlinear equation in the form

$$\boxed{P(t) = A(t)\, B^{-1}(t)}$$

where $A(t)$ and $B(t)$ satisfy the linear differential eqs.

Note that

$$B(t)\, B^{-1}(t) = I$$

$$\Rightarrow \quad \dot{B}(t)\, B^{-1}(t) + B(t)\, \dot{B}^{-1}(t) = 0 \Rightarrow \dot{B}^{-1}(t) = -B^{-1}(t)\, \dot{B}(t)\, B(t)$$

$$(1)\ \frac{d}{dt} P(t) = \frac{d}{dt}\left(A(t) B^{-1}(t)\right) = \dot{A}(t) B^{-1}(t) + A(t)\, \dot{B}^{-1}(t)$$

$$= \dot{A}(t) B^{-1}(t) - A(t) B^{-1}(t)\, \dot{B}(t)\, B(t)$$

$$(2)\ \frac{d}{dt} P(t) = F(t)P(t) + P(t)F^T(t) + Q(t) - P(t)H^T(t)R^{-1}(t)H(t)P(t)$$

$$= FAB^{-1} + AB^{-1}F^T + Q - AB^{-1} \cdot H^T R^{-1} H A B^{-1}$$

$(1) = (2) \Rightarrow$

$$\dot{A}B^{-1} - AB^{-1}\dot{B}B^{-1} = FAB^{-1} + AB^{-1}F^T + Q - AB^{-1}H^T R^{-1} H A B^{-1} \quad / B$$

$$\dot{A} - AB^{-1}\dot{B} = \underbrace{FA + QB} + AB^{-1}(F^T B - H^T R^{-1} H A)$$

$$\begin{aligned} \dot{A} &= FA + QB \\ \dot{B} &= -(F^T B - H^T R^{-1} H A) \end{aligned} \Rightarrow \begin{bmatrix} \dot{A} \\ \dot{B} \end{bmatrix} = \underbrace{\begin{bmatrix} F & Q \\ H^T R H & -F^T \end{bmatrix}}_{\text{Hamiltonian matrix}} \begin{bmatrix} A \\ B \end{bmatrix}$$

$\Rightarrow$ Hamiltonian method for solving diff. Riccati

$$\begin{bmatrix} \dot{A}(t) \\ \dot{B}(t) \end{bmatrix} = \begin{bmatrix} F(t) & Q(t) \\ H^T(t)R(t)H(t) & -F^T(t) \end{bmatrix} \begin{bmatrix} A(t) \\ B(t) \end{bmatrix} \quad (*)$$

$$P(t) = A(t)B^{-1}(t)$$

Note that $P(t_0) = P_0$ and that $B(t)$ must be invertible for all t. Since

$$P(t_0) = P_0 = A(t_0)B^{-1}(t_0)$$

take $\underline{B(t_0) = I \text{ and } A(t_0) = P_0}$

(*) can be solved only numerically since we have a time varying system.

For time invariant systems

$$\begin{bmatrix} A(t) \\ B(t) \end{bmatrix} = e^{[\quad](t-t_0)} \begin{bmatrix} P_0 \\ I \end{bmatrix}$$

$[\qquad]$ is the Hamiltonian matrix which has the property that its eigenvalues are symmetrically distributed with respect to the imaginary axis.

$\cancel{\boxed{4.8.3}}$ Scalar case

$\cancel{\boxed{4.8.4}}$ " "

$\cancel{\boxed{4.8.5}}$ " "

$\cancel{\boxed{4.8.6}}$ " "

$\boxed{4.8.7}$ The Algebraic Riccati equation solved by the Newton method

$$\left.\begin{aligned} &f(x) = 0 \\ &x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \end{aligned}\right\}$$

let x_k be known and expand $f(x_k + \Delta x)$ into Taylor series

$$f(x_k) + \frac{df}{dx}\Delta x + \frac{d^2f}{dx^2}\Delta x^2 + \cdots = 0$$

Assume Δx small (x_k good initial guess)

Newton Method $\Leftarrow$ $f(x_k) + \left.\frac{df}{dx}\right|_{x_k} (x_{k+1} - x_k) = 0$

Do not read 4.8.7 from the book. There is a very simple way to develop the Newton Method for the Algebraic Riccati Equation

$$PF^T + FP + Q - PSP = 0$$

$$(P_k+\Delta P)F^T + F(P_k+\Delta P) + Q - (P_k+\Delta P)S(P_k+\Delta P) = 0$$

We have assumed that a good initial guess P_k is known. Neglect $(\Delta P)^2$ term and denote

$P_k + \Delta P = P_{k+1}$

$$P_{k+1}F^T + FP_{k+1} + Q - P_kSP_k - P_kS\Delta P - \Delta P S P_k - \underbrace{\cancel{\Delta P S \Delta P}}_{\text{neglected}} = 0$$

$$P_{k+1}F^T + FP_{k+1} + Q - P_kS(P_k+\Delta P) - \Delta P S P_k \pm P_kSP_k = 0$$

$$P_{k+1}F^T + FP_{k+1} + Q + P_kSP_k - P_kSP_{k+1} - P_{k+1}SP_k = 0$$

$$\boxed{P_{k+1}(F-P_kS)^T + (F-P_kS)P_{k+1} + Q + P_kSP_k = 0}$$

We got the algebraic Lyapunov equation that has to be solved iteratively. If P_k is a good initial guess that the algorithm converges in 3-4 iterations (Newton method is known for its quadratic rate of convergence).

Note that P_k must stabilize $(F-P_kS)$

Kleinman (1968) has shown that assuming $F-P_kS$ is stable, it remains stable for every k

Stability of $(F-P_kS)$ $\Rightarrow$ a unique solution to the above Lyapunov equation

4.8.8 MacFarlane-Potter-Fath Eigenvector Method for Solving the Algebraic Riccati Equation

Let $v_1, v_2, \ldots v_n, v_{n+1}, \ldots, v_{2n}$ be the eigenvectors of

$$\begin{bmatrix} F & Q \\ H^T R^{-1} H & -F^T \end{bmatrix} = \Psi_c$$

Take any combination of n eigenvectors, that is form the matrix

$$\begin{bmatrix} A \\ B \end{bmatrix}^{2n \times n}$$

Assuming that B is invertible, then

$$P = A B^{-1}$$

is a solution of

$$PF^T + FP + Q - PH^T R^{-1} H P = 0$$

Since the Hamiltonian matrix $[\Psi_c]$ has n-stable and n-unstable eigenvalues by using

$$\begin{bmatrix} A \\ B \end{bmatrix} = [\text{stable eigenvector subspace}]$$

corresponding to stable eigenvalues

we get that

$$F - P H^T R^{-1} H$$

is a stable matrix

<u>Lemma 1</u> (page 135). If A and B are $n \times n$ matrices such that B is nonsingular and

$$\Psi_c \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix} D$$

for any $n \times n$ matrix D. Then $P = AB^{-1}$ satisfies the algebraic Riccati equation.

Proof:

$$\begin{bmatrix} F & Q \\ H^T R^{-1} H & -F^T \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} AD \\ BD \end{bmatrix}$$

$\Rightarrow$

$$FA + QB = AD \quad / B^{-1}$$

$$AB^{-1} / \quad H^T R^{-1} H A - F^T B = BD \quad / B^{-1}$$

$$FAB^{-1} + Q = ADB^{-1}$$

$$AB^{-1} H^T R^{-1} H A B^{-1} - A B^{-1} F^T = ADB^{-1}$$

$$\Rightarrow \quad F\underbrace{AB^{-1}}_{P} + Q = \underbrace{AB^{-1}}_{P} H^T R^{-1} H \underbrace{AB^{-1}}_{P} - \underbrace{AB^{-1}}_{P} F^T$$

or
$$FP + Q = PH^T R^{-1} H P - PF^T$$

$$FP + PF^T + Q - PH^T R^{-1} H P = 0$$

Comment:

these days the Schur method (the Hamiltonian matrix is put in the Schur form) developed by A. Laub in 1979 is considered as the most efficient method for solving the algebraic Riccati equation. It is used also in MATLAB.

4.9 Discrete-Time Riccati Equations (Difference and Algebraic ones)

$$\hat{x}_k(+) = \hat{x}_k(-) + \bar{K}_k\left(z_k - H_k\hat{x}_k(-)\right)$$

$$P_k(-) = \Phi_{k-1} P_{k-1}(+)\Phi_{k-1}^T + Q_{k-1}$$

$$P_k(+) = (I - \bar{K}_k H_k) P_k(-)$$

$$\bar{K}_k = P_k(-)H_k^T\left(H_k P_k(-) H_k^T + R_k\right)^{-1}$$

From Table 4.3 page 112

Eliminating $P_k(+)$ we get

$$P_k(-) = \Phi_{k-1}\left(I - \bar{K}_{k-1}H_{k-1}\right)P_{k-1}(-)\,\Phi_{k-1}^T + Q_{k-1}$$

Eliminating $\bar{K}_{k-1}$ produces

$$P_k(-) = \Phi_{k-1}\left[I - P_{k-1}(-)H_{k-1}^T\left(H_{k-1}P_{k-1}(-)H_{k-1}^T + R_{k-1}\right)^{-1}H_{k-1}\right]P_{k-1}(-)\Phi_{k-1}^T + Q_{k-1}$$

$$\boxed{\begin{aligned}P_k(-) = {} & \Phi_{k-1}P_{k-1}(-)\Phi_{k-1}^T + Q_{k-1} \\ & - \Phi_{k-1}P_{k-1}(-)H_{k-1}^T\left(H_{k-1}P_{k-1}(-)H_{k-1}^T + R_{k-1}\right)^{-1}H_{k-1}P_{k-1}(-)\Phi_{k-1}^T\end{aligned}}$$

Difference Riccati equation

If the matrices Φ, Q, H, R are constant at steady state

$$P_k(-) = P_{k-1}(-) = P$$

so that we get

$$\boxed{P = \Phi P \Phi^T + Q - \Phi P H^T\left(H P H^T + R\right)^{-1} H P \Phi^T}$$

Discrete-time Algebraic Riccati Equation

(4.9.2) Solution of the Difference Riccati Equation

Lemma 2 Assume that Φ_k is nonsingular, let

$$P_k(-) = A_k B_k^{-1}$$

then

$$P_{k+1}(-) = A_{k+1} B_{k+1}^{-1}$$

where

$$\begin{bmatrix} A_{k+1} \\ B_{k+1} \end{bmatrix} = \underbrace{\begin{bmatrix} \Phi_k + Q_k \Phi_k^{-T} H_k^T R_k^{-1} H_k & Q_k \Phi_k^{-T} \\ \Phi_k H_k^T R_k^{-1} H_k & \Phi_k^{-T} \end{bmatrix}}_{\Psi_d} \begin{bmatrix} A_k \\ B_k \end{bmatrix}$$

Proof: no

~~(4.9.3)~~ Scalar case

(4.9.4) MacFarlane-Potter-Fath Eigenvector Approach

Lemma 3 If A and B are $n \times n$ matrices such that B is nonsingular and

$$\Psi_d \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix} D$$

for any $n \times n$ nonsingular matrix D, then

$$P = A B^{-1}$$

satisfies the algebraic discrete-time Riccati equation

$$P = \phi P \phi - \phi P H^T (H P H^T + R)^{-1} H P \phi^T + Q$$

Proof: no

Method: Find the eigenvectors of Ψ_d corresponding to stable eigenvalues. Use these n vectors to form

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \text{stable e.v.} \end{bmatrix}$$

then

$$P = A B^{-1}$$