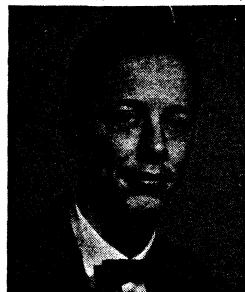


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## A Direct Derivation of the Optimal Linear Filter Using the Maximum Principle

MICHAEL ATHANS, MEMBER, IEEE, AND EDISON TSE

**Abstract**—The purpose of this paper is to present an alternate derivation of optimal linear filters. The basic technique is the use of a matrix version of the maximum principle of Pontryagin coupled with the use of gradient matrices to derive the optimal values of the filter coefficients for minimum variance estimation under the requirement that the estimates be unbiased. The optimal filter which is derived turns out to be identical to the well-known Kalman-Bucy filter.

### I. INTRODUCTION

THE CLASSIC paper by Kalman and Bucy<sup>[1]</sup> has produced one of the most useful theoretical and computationally feasible approaches to practical problems of estimation, filtering, and prediction. The so-called Kalman-Bucy filter has been used in many applications in aerospace-related problems.

The purpose of this paper is to provide another technique for obtaining optimal linear filters; it can also be

viewed as an alternate derivation of the Kalman-Bucy filter. The original derivation<sup>[1]</sup> was based upon the derivation of the Wiener-Hopf equation using the orthogonal projection lemma; the resultant integral equations were then transformed into differential equations. The method used here is conceptually and mathematically different. It requires the use of the maximum (or minimum) principle of Pontryagin and is based upon viewing the filter as a dynamical system which contains integrators and gains in forward and feedback loops. The optimal filter is then specified by 1) fixing its structure, and 2) fixing the gains.

Certainly, this paper can be viewed as an intellectual exercise. After all, no new results are presented. However, the method of attack seems promising—in the opinion of the authors—for attacking suboptimal linear and nonlinear filtering problems (see Tse<sup>[2]</sup>).

The problem is formulated in such a way that, on the basis of the dynamic behavior of the error covariance matrix, the gains involved in the structure of the optimal filter can be found which will minimize a scalar function of the error covariance matrix. Thus, by analogy to a conventional optimal control problem, 1) the elements of the error covariance matrix resemble the

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"state variables" of a dynamical system, and 2) the gains of the linear filter resemble the "control variables" of the dynamical system. This approach leads then to a natural application of the maximum principle. It should be remarked that this point of view has been successfully used for the determination of optimal radar waveforms in communication problems.<sup>[4], [5]</sup>

The structure of the paper is as follows. In Section II the plant whose state  $x(t)$  is to be estimated is defined; it is a linear time-varying plant driven by white noise, and it produces an observed signal which consists of the plant output corrupted by additive white noise (Fig. 1). In Section III the purpose and structure of the filter are discussed. The purpose of any estimating filter is to generate an estimate of the state of the plant. It is argued that a reasonable choice is to use a linear filter of the same dimension as that of the plant; this choice fixes the structure of the filter (Fig. 2). Thus, the filter is completely specified by 1) its initial state  $w(t_0)$ , 2) the gains in the forward paths, i.e., the elements of the matrix  $G(t)$ , and 3) the gains in the feedback paths, i.e., the elements of the matrix  $F(t)$ . In Section IV the requirement that the filter generate unbiased estimates of the plant state is imposed. This requirement is shown to 1) fix the initial filter state, and 2) yield a relation between the filter gain matrices  $F(t)$  and  $G(t)$ . In this manner, only the class of filters is considered that yields unbiased estimates (Fig. 3). This class of filters is parameterized by the matrix  $G(t)$ . In Section V the matrix differential equation satisfied by the error covariance matrix  $\Sigma(t)$  is obtained; this equation involves the matrix  $G(t)$ . In Section VI it is shown how the application of the minimum principle yields the optimal value of the matrix  $G(t)$  when the expected value of a quadratic form in the error is to be minimized. Section VII contains the discussion of the results.

## II. DEFINITION OF THE PLANT

In this section the plant and the filter are defined using their input-output state representation.

Consider an  $n$ th-order linear and time-varying dynamical system  $\mathcal{P}$ , called the plant (Fig. 1). The plant is described by the relations

$$\mathcal{P}: \quad \dot{x}(t) = A(t)x(t) + B(t)u(t) \quad (1)$$

$$y(t) = C(t)x(t) \quad (2)$$

where

- $x(t)$  is the state of  $\mathcal{P}$ , a vector with  $n$  components,
- $u(t)$  is the input of  $\mathcal{P}$ , a vector with  $r$  components,
- $y(t)$  is the output of  $\mathcal{P}$ , a vector with  $m$  components,
- $A(t)$  is the  $n \times n$  system matrix of  $\mathcal{P}$ ,
- $B(t)$  is the  $n \times r$  gain matrix of  $\mathcal{P}$ ,
- $C(t)$  is the  $m \times n$  output matrix of  $\mathcal{P}$ .

It is assumed that  $\mathcal{P}$  is uniformly completely controllable and observable.

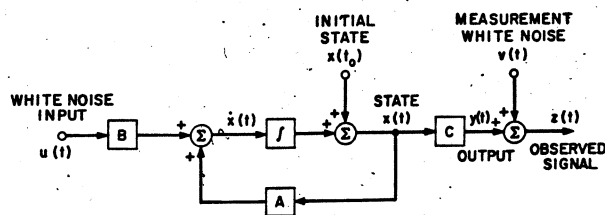


Fig. 1. Block diagram of the plant  $\mathcal{P}$  which results in the observed signal  $z(t)$ . All matrices  $A$ ,  $B$ , and  $C$  are time varying.

It is assumed that the input  $u(t)$  to  $\mathcal{P}$  is a vector-valued white-noise Gaussian process with zero mean

$$E\{u(t)\} = 0 \quad \text{for all } t \quad (3)$$

and covariance matrix

$$\text{cov}[u(t); u(\tau)] = E\{u(t)u'(\tau)\} = \delta(t - \tau)Q(t) \quad (4)$$

where  $\delta(\cdot)$  is the Dirac delta function. Evidently,  $Q(t)$  is an  $r \times r$  symmetric positive semidefinite matrix.

Let  $t_0$  denote the initial time and  $x(t_0)$  the initial state vector of  $\mathcal{P}$ . It is assumed that  $x(t_0)$  is a vector-valued Gaussian random variable, independent of  $u(t)$ , with known mean

$$E\{x(t_0)\} \triangleq \bar{x}_0 \quad (5)$$

and known covariance matrix

$$\begin{aligned} \text{cov}[x(t_0); x(t_0)] &= E\{[x(t_0) - \bar{x}_0][x(t_0) - \bar{x}_0]'\} \\ &\triangleq \Sigma_0. \end{aligned} \quad (6)$$

Under these assumptions the state  $x(t)$  and the output  $y(t)$  are Gaussian random processes.

Suppose that the output  $y(t)$  can be observed only in the presence of white Gaussian noise. For this reason, let  $z(t)$  denote the observed signal,

$$z(t) = y(t) + v(t) = C(t)x(t) + v(t) \quad (7)$$

where  $v(t)$  is a Gaussian white-noise process with zero mean, i.e.,

$$E\{v(t)\} = 0 \quad \text{for all } t \quad (8)$$

and covariance matrix

$$\text{cov}[v(t); v(\tau)] = E\{v(t)v'(\tau)\} = \delta(t - \tau)R(t) \quad (9)$$

with  $R(t)$  symmetric positive definite.<sup>1</sup> Furthermore, it is assumed that  $v(t)$ ,  $u(t)$ , and  $x(t_0)$  are independent.

## III. THE FILTER

In the problem of estimation, it is desired to obtain an estimate of the state  $x(t)$  of the plant  $\mathcal{P}$ . The state of  $\mathcal{P}$  is not, in general, available for measurement. Rather, the signal  $z(\tau)$  can be measured for  $t_0 \leq \tau \leq t$ , and, on the basis of this an estimate, say  $\hat{x}(t)$ , of  $x(t)$  can be obtained.

<sup>1</sup> Unless this assumption is made, a singular optimization problem will be involved.

From an engineering point of view, a system  $\mathcal{F}$ , called a filter, must be constructed to accept the available data in real time, namely,  $z(\cdot)$ , and produce a vector-valued signal  $\hat{x}(t)$  for  $t \geq t_0$  such that the error signal

$$e(t) = x(t) - \hat{x}(t) \quad (10)$$

is in some sense small.

From a practical point of view, the following two questions are of interest:

- 1) How can a filter  $\mathcal{F}$  be constructed and how complex is its structure?
- 2) How good is the filter (in the sense of estimating the state  $x(t)$ )?

The complexity of the filter can be related to several of its properties; for example, whether or not the filter is continuous-time or sampled-data, lumped or distributed, linear or nonlinear, time varying or time invariant, and so on. In general, the physical complexity of the filter is also related to the complexity of the mathematics that describe the plant-filter process.

As a general comment, two types of design should be distinguished. One type is to seek the best possible filter without any additional constraints. On the other hand, additional constraints (motivated by practical, mathematical, or computational considerations) may be imposed before the optimization problem is formulated. In the latter case, a filter may be obtained which is optimal with respect to the imposed constraints, but not identical to the truly optimal one. However, in that case the design may be more in accord with the requirements of the designer. In the sequel, the philosophy of design will be to constrain the filter to be a linear one. Of course, this constraint turns out to be of no consequence in this case of linear dynamics and Gaussian processes, since the Kalman-Bucy filter is also linear; in general, the results will be different.

The filter  $\mathcal{F}$  is constrained to be a linear and time-varying system described by the relations

$$\dot{w}(t) = F(t)w(t) + G(t)z(t) \quad (11)$$

$$\hat{x}(t) = H(t)w(t) \quad (12)$$

where  $\hat{x}(t)$  is an  $n$  vector (like  $x(t)$ ) and  $z(t)$  an  $m$  vector. Note that the complexity of this linear filter can be related to the dimension, say  $q$ , of its state vector  $w(t)$ . It may be argued that the smaller the value of  $q$ , the less complex the filter. In the remainder of this paper, this class of linear filters is further restricted by<sup>2</sup>

$$q = n;$$

$$H(t) = I \quad (13)$$

where  $I$  is the  $n \times n$  identity matrix, so that the filter is described by the equation

<sup>2</sup> This assumption will be discussed in the sequel.

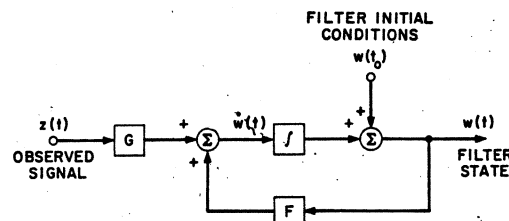


Fig. 2. Dynamic simulation of the filter.

$$\mathcal{F}: \dot{w}(t) = F(t)w(t) + G(t)z(t) \quad (14)$$

where the state  $w(t)$  of the filter  $\mathcal{F}$  is to act as the estimate of the state  $x(t)$  of the plant  $\mathcal{P}$ .

Once the structure of the filter  $\mathcal{F}$  (Fig. 2) has been constrained by (14), then the specification of the elements  $f_{ij}(t)$  of the  $n \times n$  matrix  $F(t)$ , the elements  $g_{ik}(t)$  of the  $n \times m$  matrix  $G(t)$ , and the initial state vector (deterministic)  $w(t_0)$  completely defines the filter  $\mathcal{F}$  in the sense that  $w(t)$  is well defined for any  $z(\tau)$ ,  $t_0 \leq \tau \leq t$ .

In the remainder of this paper the initial state  $w(t_0)$  and the matrices  $F(t)$  and  $G(t)$  of the filter will be determined by demanding that 1)  $w(t)$  be an unbiased estimate of  $x(t)$ , and then 2)  $w(t)$  be a minimum variance estimate of  $x(t)$ .

#### IV. STRUCTURE OF THE FILTER FOR UNBIASED ESTIMATION

The requirement that  $w(t)$  be an unbiased estimate of  $x(t)$  is basic to the subsequent development. This requirement of unbiasedness may not be imposed (or even desirable) in a statistical estimator, although it is a property of the conditional expectation. Nevertheless, such a constraint appears to be popular and desirable in many engineering applications of the theory, and it is made in this paper.

Since  $w(t)$  is to be an estimate of  $x(t)$ , consider the error vector

$$e(t) = x(t) - w(t). \quad (15)$$

Differentiating formally both sides of (15) and using (1), (2), (7), and (14), it may be concluded that the error  $e(t)$  satisfies the differential equation

$$\begin{aligned} \dot{e}(t) &= A(t)x(t) + B(t)u(t) - F(t)w(t) \\ &\quad - G(t)C(t)x(t) - G(t)v(t). \end{aligned} \quad (16)$$

Since  $w(t) = x(t) - e(t)$ , (16) further reduces to

$$\begin{aligned} \dot{e}(t) &= [A(t) - F(t) - G(t)C(t)]x(t) + F(t)e(t) \\ &\quad + B(t)u(t) - G(t)v(t). \end{aligned} \quad (17)$$

Next, impose the requirement that  $w(t)$  be an unbiased estimate of  $x(t)$  for all  $t \geq t_0$ ; this means that the constraint

$$E\{x(t)\} = E\{w(t)\} \quad \text{for all } t \geq t_0 \quad (18)$$

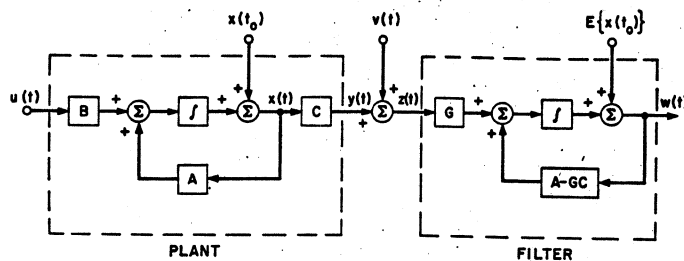


Fig. 3.  $w(t)$  is an unbiased estimate of  $x(t)$  for all  $t \geq t_0$  for any matrix  $G(t)$ . All matrices are time varying.

must hold. The constraining equation (18) yields

$$E\{e(t)\} = 0 \quad \text{for all } t \geq t_0 \quad (19)$$

$$\frac{d}{dt} E\{e(t)\} = E\{\dot{e}(t)\} = 0 \quad \text{for all } t \geq t_0. \quad (20)$$

By taking the expectation of both sides of (17), recalling that  $E\{u(t)\} = 0$  and  $E\{v(t)\} = 0$ , and using (19) and (20), it may be deduced that the equality

$$[A(t) - F(t) - G(t)C(t)]E\{x(t)\} = 0 \quad (21)$$

must hold for all  $t \geq t_0$ . But, in general,  $E\{x(t)\} \neq 0$  since it is the solution of the differential equation (see Appendix I)

$$\begin{aligned} \frac{d}{dt} E\{x(t)\} &= A(t)E\{x(t)\}; \\ E\{x(t_0)\} &= \bar{x}_0. \end{aligned} \quad (22)$$

Therefore, if (21) is to hold for all  $E\{x(t)\}$ , then one arrives at the relation

$$A(t) - F(t) - G(t)C(t) = 0 \quad \text{for all } t \geq t_0. \quad (23)$$

Furthermore, from (15) the requirement that at the initial time  $E\{e(t_0)\} = 0$  yields (see (5))

$$w(t_0) = E\{x(t_0)\} \triangleq \bar{x}_0 \quad (24)$$

since  $w(t_0)$  is a deterministic vector (the initial state of the filter).

**Recapitulation:** The constraint that  $w(t)$  be an unbiased estimate of  $x(t)$  for all  $t \geq t_0$  implies that

- 1) the initial filter state  $w(t_0)$  must be equal to the mean of the initial plant state, according to (24), and
- 2) the filter matrices  $F(t)$  and  $G(t)$  must be related, in view of (23), by

$$F(t) = A(t) - G(t)C(t). \quad (25)$$

Henceforth, only the class of "unbiased filters"  $\mathcal{F}_u$  described by the relation

$$\begin{aligned} \dot{w}(t) &= [A(t) - G(t)C(t)]w(t) + G(t)z(t); \\ \mathcal{F}_u: \quad w(t_0) &= \bar{x}_0 \end{aligned} \quad (26)$$

will be considered. This class of filters  $\mathcal{F}_u$  contains as a parameter the "gain" matrix  $G(t)$  (Fig. 3).

## V. TIME EVOLUTION OF THE ERROR COVARIANCE MATRIX

Consider the class of all linear filters  $\mathcal{F}_u$  which provide unbiased estimates  $w(t)$  of  $x(t)$ . Such filters are described by (26) where  $G(t)$  is an arbitrary  $n \times m$  matrix. For any given choice of  $G(t)$  the error  $e(t)$  satisfies the differential equation

$$\dot{e}(t) = [A(t) - G(t)C(t)]e(t) + B(t)u(t) - G(t)v(t) \quad (27)$$

which is obtained by substituting (25) into (17).

From a physical point of view it is desirable to have an unbiased estimate with small variance. For this reason, let  $\tau$  be some time  $\tau > t_0$ , and consider the scalar quantity

$$\begin{aligned} J &= E\{e'(\tau)M(\tau)e(\tau)\}; \\ M(\tau) &= M'(\tau) \end{aligned} \quad (28)$$

where the matrix  $M(\tau)$  is constrained to be positive definite.<sup>3</sup> Since for any two column vectors  $x$  and  $y$  the following equality holds

$$x'y = \text{tr}[yx'] \quad (29)$$

it follows that

$$J = E\{\text{tr}[M(\tau)e(\tau)e'(\tau)]\} = \text{tr}[M(\tau)E\{e(\tau)e'(\tau)\}] \quad (30)$$

because the trace and expectation operators are linear and they commute. But the matrix

$$\Sigma(\tau) \triangleq E\{e(\tau)e'(\tau)\} \quad (31)$$

is simply the covariance matrix of the error  $e(\tau)$ , i.e.,

$$\Sigma(\tau) = \text{cov}[e(\tau); e(\tau)]$$

since  $E\{e(t)\} = 0$  for all  $t$ . Henceforth, the scalar quantity

$$J = \text{tr}[M(\tau)\Sigma(\tau)] \quad (32)$$

shall be used as a measure of the filter performance. The smaller the value of  $J$ , the better the filter.

Since the quality of the filter can be related to a function of the error covariance matrix  $\Sigma(\cdot)$ , consider its time evolution. As the error differential equation (27) is linear, the results stated in Appendix I can be used to deduce that  $\Sigma(t)$  satisfies the linear matrix differential equation

$$\begin{aligned} \dot{\Sigma}(t) &= [A(t) - G(t)C(t)]\Sigma(t) + \Sigma(t)[A(t) - G(t)C(t)]' \\ &\quad + B(t)Q(t)B'(t) + G(t)R(t)G'(t). \end{aligned} \quad (33)$$

Furthermore, the value of  $\Sigma(t_0)$  is known to be

$$\begin{aligned} \Sigma(t_0) &= E\{e(t_0)e'(t_0)\} \\ &= E\{[x(t_0) - w(t_0)][x(t_0) - w(t_0)]'\} \\ &= E\{[x(t_0) - \bar{x}_0][x(t_0) - \bar{x}_0]'\} = \Sigma_0 \end{aligned} \quad (34)$$

<sup>3</sup> The matrix  $M(\tau)$  is included so that there is freedom to penalize certain components of the error vector more heavily than others. However, it turns out that the optimal filter is independent of  $M(\tau)$ .

in view of (24) and (6). In other words, the initial value of the error covariance matrix equals the covariance matrix of the initial plant state.

Clearly, for any given matrix  $G(t)$ , (33) and (34) completely specify the error covariance matrix  $\Sigma(t)$  for all  $t \geq t_0$ . Thus, the scalar quantity  $J$  of (32) can be evaluated. If it is assumed that the plant matrices  $A(t)$ ,  $B(t)$ ,  $C(t)$  and the covariance matrices  $Q(t)$ ,  $R(t)$ , and  $\Sigma_0$  are given, then the value of  $J$  will depend upon the choice of the filter matrix  $G(t)$  for  $t_0 \leq t \leq \tau$ .

## VI. OPTIMAL FILTER

The preceding discussion leads naturally to the question: How can the matrix  $G(t)$ ,  $t_0 \leq t \leq \tau$ , be chosen so as to minimize the "cost functional"  $J$ ? The answer to this question involves the solution of a deterministic optimization problem. The precise formulation of the problem is as follows.

Given: The matrix differential equation satisfied by the error covariance matrix  $\Sigma(t)$  (see (33))

$$\begin{aligned} \dot{\Sigma}(t) &= [A(t) - G(t)C(t)]\Sigma(t) + \Sigma(t)[A(t) - G(t)C(t)]' \\ &\quad + B(t)Q(t)B'(t) + G(t)R(t)G'(t); \\ \Sigma(t_0) &= \Sigma_0, \end{aligned} \quad (35)$$

a terminal time  $\tau$ , and the cost functional

$$J = \text{tr}[M(\tau)\Sigma(\tau)]. \quad (36)$$

Determine: The matrix  $G(t)$ ,  $t_0 \leq t \leq \tau$ , so as to minimize the cost functional (36).

The elements  $\sigma_{ij}(t)$  of  $\Sigma(t)$  may be considered as the "state variables" of a system, and the elements  $g_{ik}(t)$  of  $G(t)$  as the "control variables" in an optimal control problem. The cost functional is then a terminal-time penalty function on the state variables  $\sigma_{ij}(\tau)$ . Thus the minimum principle of Pontryagin can be applied to determine the optimal matrix  $G(t)$ ,  $t_0 \leq t \leq \tau$ .

To do this, define a set of costate variables  $p_{ij}(t)$  corresponding to the  $\sigma_{ij}(t)$ ,  $i, j = 1, 2, \dots, n$ . Define an  $n \times m$  costate matrix  $P(t)$ , associated with the matrix  $\Sigma(t)$ , so that the  $ij$ th element of  $P(t)$  is given by  $p_{ij}(t)$ . Then the minimum principle<sup>[8]</sup> can be used. This necessitates the definition of the Hamiltonian function  $H$  given by

$$H = \sum_{i,j=1}^n \dot{\sigma}_{ij}(t) p_{ij}(t) = \text{tr}[\dot{\Sigma}(t)P'(t)]. \quad (37)$$

Substituting (35) into (37),

$$\begin{aligned} H &= \text{tr}[A(t)\Sigma(t)P'(t)] - \text{tr}[G(t)C(t)\Sigma(t)P'(t)] \\ &\quad + \text{tr}[\Sigma(t)A'(t)P'(t)] - \text{tr}[\Sigma(t)C'(t)G'(t)P'(t)] \\ &\quad + \text{tr}[B(t)Q(t)B'(t)P'(t)] + \text{tr}[G(t)R(t)G'(t)P'(t)]. \end{aligned} \quad (38)$$

The Hamiltonian function  $H$  is quadratic in the elements  $g_{ik}(t)$  of the matrix  $G(t)$ . Thus, from the minimum principle, the necessary condition

$$\left. \frac{\partial H}{\partial g_{ik}(t)} \right|_* = 0 \quad (39)$$

is obtained, where  $*$  is used to indicate optimal quantities and indicates that the quantity above must be evaluated along the optimal.

According to the minimum principle, the costate variables  $p_{ij}(t)$  must satisfy the differential equations

$$\dot{p}_{ij}^*(t) = - \left. \frac{\partial H}{\partial \sigma_{ij}(t)} \right|_*. \quad (40)$$

Furthermore, the transversality conditions at the terminal time  $\tau$  are

$$p_{ij}^*(\tau) = \left. \frac{\partial}{\partial \sigma_{ij}(\tau)} \text{tr}[M(\tau)\Sigma(\tau)] \right|_*. \quad (41)$$

Using the concept of a gradient matrix (see Appendix II), (39) through (41) reduce to

$$\left. \frac{\partial H}{\partial G(t)} \right|_* = 0 \quad (42)$$

$$\dot{P}^*(t) = - \left. \frac{\partial H}{\partial \Sigma(t)} \right|_* \quad (43)$$

$$P^*(\tau) = \left. \frac{\partial}{\partial \Sigma(\tau)} \text{tr}[M(\tau)\Sigma(\tau)] \right|_*. \quad (44)$$

Using the formulas of Appendix II it is found that (42) yields the relationship

$$\begin{aligned} -P^*(t)\Sigma^{*'}(t)C'(t) - P^{*'}(t)\Sigma^*(t)C'(t) + P^*(t)G^*(t)R'(t) \\ + P^{*'}(t)G^*(t)R(t) = 0. \end{aligned} \quad (45)$$

Equation (43) yields the matrix differential equation

$$\begin{aligned} \dot{P}^*(t) &= -A'(t)P^*(t) - C'(t)G^{*'}(t)P^*(t) - P^*(t)A(t) \\ &\quad - P^*(t)G^*(t)C(t) \end{aligned} \quad (46)$$

$$\begin{aligned} &= -P^*(t)[A(t) - G^*(t)C(t)] \\ &\quad - [A(t) - G^*(t)C(t)]'P^*(t). \end{aligned} \quad (47)$$

Finally, (44) yields

$$P^*(\tau) = M(\tau). \quad (48)$$

Equation (47) is a linear matrix differential equation in  $P^*(t)$ ; this fact, coupled with the fact that  $P^*(\tau)$  is symmetric and positive definite, implies that<sup>4</sup>

$$P^*(t) \text{ is symmetric positive definite.} \quad (49)$$

Thus  $P^*(t) = P^{*'}(t)$ , and moreover  $[P^*(t)]^{-1}$  exists; this means that (45) reduces to

$$2G^*(t)R(t) = \Sigma^{*'}(t)C'(t) + \Sigma^*(t)C'(t) \quad (50)$$

since  $R(t)$  is assumed symmetric positive definite. But

<sup>4</sup> This follows trivially from the general form of the solution of a matrix differential equation (see Bellman,<sup>[10]</sup> p. 175).

the error covariance matrix  $\Sigma^*(t)$  is also symmetric and, therefore, the optimal value for  $G^*(t)$  must be given by

$$G^*(t) = \Sigma^*(t)C'(t)R^{-1}(t). \quad (51)$$

The matrix  $\Sigma^*(t)$  is the error covariance matrix for the optimal filter. It is determined by substituting (51) into (35) to find

$$\begin{aligned} \dot{\Sigma}^*(t) = & [A(t) - \Sigma^*(t)C'(t)R^{-1}(t)C(t)]\Sigma^*(t) \\ & + \Sigma^*(t)[A'(t) - C'(t)R^{-1}(t)C(t)\Sigma^*(t)] \\ & + B(t)Q(t)B'(t) \\ & + \Sigma^*(t)C'(t)R^{-1}(t)R(t)R^{-1}(t)C(t)\Sigma^*(t) \end{aligned} \quad (52)$$

which reduces to

$$\begin{aligned} \dot{\Sigma}^*(t) = & A(t)\Sigma^*(t) + \Sigma^*(t)A'(t) + B(t)Q(t)B'(t) \\ & - \Sigma^*(t)C'(t)R^{-1}(t)C(t)\Sigma^*(t). \end{aligned} \quad (53)$$

Equation (53) is a matrix differential equation of the Riccati type. Recall that the initial condition, assumed known, is

$$\Sigma^*(t_0) = \Sigma_0. \quad (54)$$

Then, the error covariance matrix for the optimal unbiased minimum variance filter is completely specified by (53) and (54), and the solution  $\Sigma^*(t)$  specifies the optimal filter.

## VII. DISCUSSION OF THE RESULTS

The preceding derivations complete the determination of the optimal filter. In essence, the optimal filter is specified by the relations

### (1) Initial filter state (see (24))

$$w(t_0) = \bar{x}_0 \quad (55)$$

### (2) Feedback matrix (see (25) and (51))

$$F^*(t) = A(t) - \Sigma^*(t)C'(t)R^{-1}(t)C(t) \quad (56)$$

### (3) Forward matrix (see (51))

$$G^*(t) = \Sigma^*(t)C'(t)R^{-1}(t). \quad (57)$$

Thus, the optimal estimate  $w^*(t)$  of  $x(t)$  is generated by the system

$$\begin{aligned} \dot{w}^*(t) = & [A(t) - \Sigma^*(t)C'(t)R^{-1}(t)C(t)]w^*(t) \\ & + \Sigma^*(t)C'(t)R^{-1}(t)z(t); \\ w(t_0) = & \bar{x}_0 \end{aligned} \quad (58)$$

as illustrated in Fig. 4. The matrix  $\Sigma^*(t)$  is completely defined by the Riccati equation (53) with the boundary condition (54).

The optimal filter thus derived turns out to be identical to the Kalman-Bucy filter.<sup>[1]</sup> In the remainder of this section properties which are immediately obvious from the development contained in the main part of this paper are discussed.

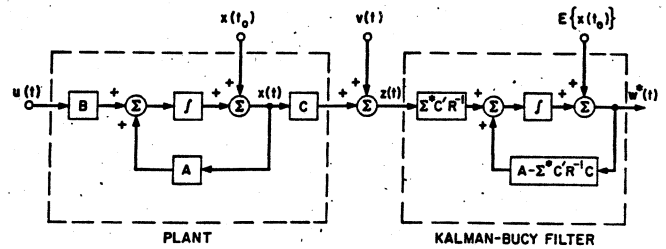


Fig. 4.  $w^*(t)$  is the unbiased minimum variance estimate of  $x(t)$ . All matrices are time varying.  $\Sigma^*(t)$  is the solution of the Riccati equation (53).

### A. Dependence on the Terminal Time $\tau$

Observe that although the optimization problem was formulated so as to minimize a weighted scalar function of the error covariance matrix at some fixed time  $\tau$ , the answer turns out to be independent of  $\tau$ . The value of the costate matrix  $P^*(t)$  does depend on  $\tau$ ; however, only the fact that the costate matrix  $P^*(t)$  is symmetric and invertible is used to eliminate  $P^*(t)$  in (45). In this manner, neither  $G^*(t)$  nor  $\Sigma^*(t)$  depend on  $\tau$ . It is, therefore, obvious that the Kalman-Bucy filter produces, uniformly in  $t$ , unbiased minimum variance estimates as it accepts the observations  $z(\cdot)$  in real time.

### B. Dependence on the Cost Functional

Kalman and Bucy<sup>[1]</sup> sought a filter to minimize the expected squared error in estimating any linear function of the message. It is indeed known that the Kalman-Bucy filter is optimal for a variety of performance criteria. Similar results can be obtained using the proof presented. It is easy to see that the same conclusion would hold for the class of cost functionals which have the property that  $P^*(\tau)$  is symmetric positive definite. In other words, it is possible to consider directly cost functionals of the form

$$J_2 = g[\Sigma(\tau)] \quad (59)$$

where  $g(\cdot)$  is a scalar function of the matrix  $\Sigma(\tau)$ , if the gradient matrix

$$\left. \frac{\partial}{\partial \Sigma(\tau)} g[\Sigma(\tau)] \right|_* \quad (60)$$

is symmetric and positive definite. In this category falls the cost functional

$$J_3 = \det[\Sigma(\tau)] \quad (61)$$

because its gradient matrix

$$\left. \frac{\partial}{\partial \Sigma(\tau)} \det[\Sigma(\tau)] \right|_* = \det[\Sigma^*(\tau)][\Sigma^{*-1}(\tau)]' \quad (62)$$

is symmetric and positive definite. In fact, the cost functional (61) is related to the volume of the "error ellipsoid."<sup>[12]</sup> In short, the derived linear filter is optimal for a large class of cost functionals as long as the costate matrix  $P^*(t)$  turns out to be symmetric and nonsingular for all  $t$ ,  $t_0 \leq t \leq \tau$ .

### C. Order of the Filter

In Section IV the assumption was made that the order of the filter is the same as the order of the plant. This assumption will now be discussed.

Suppose that the filter is described by (11) and (12). The error is

$$e(t) = x(t) - \hat{x}(t) = x(t) - H(t)w(t). \quad (63)$$

If the requirement is imposed that  $\hat{x}(t)$  be an unbiased estimate of  $x(t)$  for all  $t \geq t_0$ , then at  $t = t_0$  the relation

$$H(t_0)w(t_0) = E\{x(t_0)\} \triangleq \bar{x}_0 \quad (64)$$

must hold. Thus, to specify uniquely the filter initial condition  $w(t_0)$ , the matrix  $H(t_0)$  must have rank  $n$ . This means that the order of the filter must be at least  $n$ , i.e., the dimension of the filter state vector  $w(t)$  must be at least  $n$ . Since the filter is less complex if the dimension of  $w(t)$  is small, it is reasonable to investigate (as it was already done in the main part of this paper) the case that the plant and the filter have the same dimension.

### D. Sufficiency Considerations

It is well known that the maximum principle provides necessary conditions for optimality. In this case the necessary conditions are also sufficient. This can be shown by proving that the Hamilton-Jacobi equation is satisfied for all  $\Sigma(t)$ .<sup>[8]</sup> However, the proof is not included here since it is straightforward but lengthy.

### E. Comments on the Variational Approach

The use of the calculus of variations and of the maximum principle is certainly not new in filtering problems.<sup>[6],[7],[11]</sup> However, the philosophy used here is to optimize directly the gains of the filter once the filter structure, dimensionality, and other properties, e.g., linearity, have been fixed by the designer. As mentioned before, this general technique will lead to a filter which may be inferior to the unconstrained optimal one. Nonetheless, this method of optimizing directly the feed-forward and feedback gains may prove useful to the designer.

## VIII. CONCLUSIONS

It has been demonstrated that the "matrix" minimum principle can be used to provide a direct constructive derivation of the well-known Kalman-Bucy filter. The method of proof also indicates that the Kalman-Bucy filter is optimal for a variety of performance criteria.

### APPENDIX I

#### MEAN AND COVARIANCE DIFFERENTIAL EQUATIONS

Consider a linear time-varying system with state vector  $x(t)$  and input vector  $u(t)$  described by the equation

$$\dot{x}(t) = A(t)x(t) + B(t)u(t). \quad (65)$$

Suppose that  $u(t)$  is a white-noise process with mean

$$\bar{u}(t) \triangleq E\{u(t)\} \quad (66)$$

and covariance matrix

$$\begin{aligned} \text{cov}[u(t); u(\tau)] &= E\{[u(t) - \bar{u}(t)][u(\tau) - \bar{u}(\tau)]'\} \\ &= \delta(t - \tau)Q(t). \end{aligned} \quad (67)$$

Suppose that the initial state  $x(t_0)$  of the system (65) is a vector-valued random variable with mean

$$\bar{x}_0 \triangleq E\{x(t_0)\} \quad (68)$$

and covariance matrix

$$\begin{aligned} \text{cov}[x(t_0); x(t_0)] &= E\{[x(t_0) - \bar{x}_0][x(t_0) - \bar{x}_0]'\} \\ &\triangleq \Sigma_0. \end{aligned} \quad (69)$$

Assume that  $x(t_0)$  and  $u(t)$  are independent.

Define the mean of  $x(t)$  by

$$\bar{x}(t) \triangleq E\{x(t)\} \quad (70)$$

and the covariance matrix of  $x(t)$  by

$$\begin{aligned} \text{cov}[x(t); x(t)] &= E\{[x(t) - \bar{x}(t)][x(t) - \bar{x}(t)]'\} \\ &\triangleq \Sigma(t). \end{aligned} \quad (71)$$

It can be shown that the mean  $\bar{x}(t)$  satisfies the linear vector differential equation

$$\begin{aligned} \frac{d}{dt} \bar{x}(t) &= A(t)\bar{x}(t) + B(t)\bar{u}(t); \\ \bar{x}(t_0) &= \bar{x}_0 \end{aligned} \quad (72)$$

and the covariance  $\Sigma(t)$  satisfies the linear matrix differential equation

$$\begin{aligned} \frac{d}{dt} \Sigma(t) &= A(t)\Sigma(t) + \Sigma(t)A'(t) + B(t)Q(t)B'(t); \\ \Sigma(t_0) &= \Sigma_0. \end{aligned} \quad (73)$$

The derivation of (72) is straightforward as it follows from the time differentiation of (70) and the use of the property that

$$\frac{d}{dt} E\{x(t)\} = E\dot{x}(t).$$

However, the derivation of (73) is slightly more formal. It has been stated by many authors (e.g., see Kalman,<sup>[13]</sup> pp. 201-202) and is given hereafter for the sake of completeness.

For simplicity assume that all random variables have zero mean, i.e.,

$$\begin{aligned} E\{x(t_0)\} &= 0; \\ E\{u(t)\} &= 0. \end{aligned} \quad (74)$$

Then, from (72) it follows that  $E\{x(t)\} = 0$ , and thus the covariance matrix  $\Sigma(t)$  is simply defined by

$$\Sigma(t) = E\{x(t)x'(t)\}. \quad (75)$$

Taking the time derivative of (75) yields

$$\dot{\Sigma}(t) = \frac{d}{dt} E\{x(t)x'(t)\} = E\{\dot{x}(t)x'(t)\} + E\{x(t)\dot{x}'(t)\}. \quad (76)$$

Substitution of (65) into (76) yields

$$\begin{aligned} \dot{\Sigma}(t) = & E\{A(t)x(t)x'(t)\} + E\{B(t)u(t)x'(t)\} \\ & + E\{x(t)x'(t)A'(t)\} + E\{x(t)u'(t)B'(t)\}. \end{aligned} \quad (77)$$

Now  $E\{x(t)u'(t)\}$  must be evaluated. Since the solution to (65) is

$$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^t \Phi(t, \tau)B(\tau)u(\tau)d\tau \quad (78)$$

where  $\Phi(t, t_0)$  is the transition matrix of  $A(t)$ ,

$$\begin{aligned} E\{x(t)u'(t)\} = & \Phi(t, t_0)E\{x(t_0)u'(t)\} \\ & + \int_{t_0}^t \Phi(t, \tau)B(\tau)E\{u(\tau)u'(t)\}d\tau. \end{aligned} \quad (79)$$

Since  $x(t_0)$  and  $u(t)$  are assumed independent,

$$E\{x(t_0)u'(t)\} = 0,$$

and so (79) reduces to

$$E\{x(t)u'(t)\} = \int_{t_0}^t \Phi(t, \tau)B(\tau)Q(\tau)\delta(t - \tau)d\tau \quad (80)$$

in view of (67) with  $\bar{u}(t) = 0$ . But, from the theory of distributions, the formula

$$\int_a^b f(x)\delta(b - x)dx = \frac{1}{2}f(b) \quad (81)$$

loosely implies that since the delta function occurs at the upper limit  $x=b$ , only "half" the impulse is weighed. The generalization of (81) to the matrix case exhibited by (80) yields

$$E\{x(t)u'(t)\} = \frac{1}{2}\Phi(t, t)B(t)Q(t) = \frac{1}{2}B(t)Q(t). \quad (82)$$

Similarly,

$$E\{u(t)x'(t)\} = \frac{1}{2}Q(t)B'(t). \quad (83)$$

Substitution of (82) and (83) into (77) yields

$$\dot{\Sigma}(t) = A(t)\Sigma(t) + \Sigma(t)A'(t) + B(t)Q(t)B'(t) \quad (84)$$

which is the desired relation.

## APPENDIX II

### GRADIENT MATRICES

In this appendix the concept of a gradient matrix is defined and some equations are presented. The basic ideas and a more complete set of calculations can be found in Athans and Schweppe.<sup>[9]</sup>

Let  $X$  be an  $n \times n$  matrix with elements  $x_{ij}$ . Let  $f(\cdot)$  be a scalar-valued function of the  $x_{ij}$

$$f(X) = f(x_{11}, \dots, x_{1n}, x_{21}, \dots, x_{2n}, \dots). \quad (85)$$

For example, a common scalar function is the trace of  $X$

$$f(X) = \text{tr}[X] = x_{11} + x_{22} + \dots + x_{nn}. \quad (86)$$

Consider the partial derivatives

$$\frac{\partial f(X)}{\partial x_{ij}}, \quad i, j = 1, 2, \dots, n. \quad (87)$$

Then the gradient matrix of  $f(X)$  with respect to the matrix  $X$  is defined to be the  $n \times n$  matrix whose  $ij$ th element is the function (87). The gradient matrix is denoted by

$$\frac{\partial f(X)}{\partial X}. \quad (88)$$

Straightforward computations yield the following relations

$$\frac{\partial}{\partial X} \text{tr}[X] = I \quad (89)$$

$$\frac{\partial}{\partial X} \text{tr}[AX] = A' \quad (90)$$

$$\frac{\partial}{\partial X} \text{tr}[AX'] = A \quad (91)$$

$$\frac{\partial}{\partial X} \text{tr}[AXBX] = A'X'B' + B'X'A' \quad (92)$$

$$\frac{\partial}{\partial X} \text{tr}[AXBX'] = A'XB' + AXB. \quad (93)$$

Additional relations can be obtained through the use of the trace identities

$$\text{tr}[AB] = \text{tr}[BA] \quad (94)$$

$$\text{tr}[AB'] = \text{tr}[BA']. \quad (95)$$

An extensive table of gradient matrices for trace and determinant functions can be found elsewhere.<sup>[9]</sup>

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## The Factorization of Discrete-Process Spectral Matrices

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**Abstract**—A technique is presented for solving the discrete version of the multidimensional Wiener-Hopf equation by spectral factorization. This equation is derived to establish a need for spectral factorization and to determine the requirements of the factors of the

spectral matrix. The method of factoring the spectral matrix of continuous systems, developed by Davis, is then extended to discrete systems. More specifically, a matrix  $H(z)$  must be found such that the matrix of the spectra of the input signals equals the product of  $H(z^{-1})$  and  $H^T(z)$ . A technique for finding this matrix is presented. The nonanticipatoriness as well as the stability of the elements of  $H(z)$  and  $H^{-1}(z)$  must be and is guaranteed. It is then shown that the solution to the discrete Wiener-Hopf equation is unique.

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