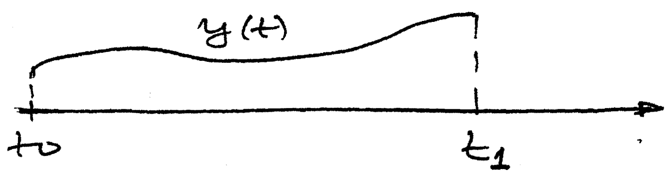
Chapter 4

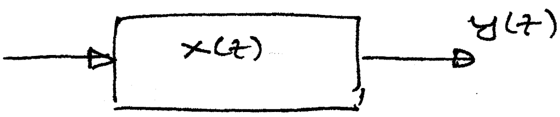
Feb. 14, 97 (4)

Linear Optimal $\underbrace{\text{Filters, Predictors \& Smoothers}}_{\text{Estimators}}$

given measurements $y(t)$



FILTERS:



Estimate $x(t_1)$ based on $y(t)$, $t_0 \le t \le t_1$

PREDICTORS:

Estimate $x(t_2)$, $t_2 > t_1$ based on $y(t)$, $t_0 \le t \le t_1$

SMOOTHERS:

Estimate $x(t)$, $\underline{t < t_1}$ based on $y(t)$, $t_0 \le t \le t_1$

Let $\hat{x}(t)$ be such an estimate of $x(t)$, then the estimation error is defined by

$$e(t) = x(t) - \hat{x}(t)$$

Our goal is to minimizes the weighted mean square error

$$\min \underset{\text{mean}}{E} \{ \underbrace{e(t)^T \overset{\text{weighted}}{M} e(t)}_{\text{squared}} \} \quad , \quad M = M^T \ge 0$$

4.2 DISCRETE-TIME KALMAN FILTER

$$x_k = \Phi_{k-1} x_{k-1} + w_{k-1} \qquad \text{system}$$

$$z_k = H_k x_k + v_k \qquad \text{measurements}$$

$$E\{w_k\} = 0, \quad E\{v_k\} = 0$$

$$E\{w_k w_i^T\} = Q_k \, \delta(k-i)$$

$$E\{v_k \cdot v_i^T\} = R_k \, \delta(k-i)$$

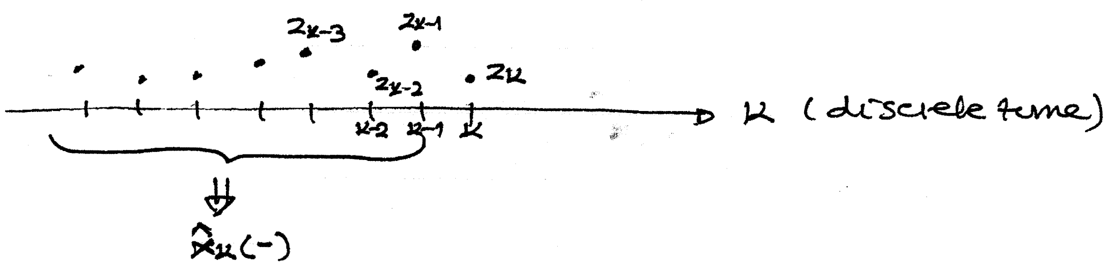
$E\{w_k x_i^T\} = 0$ and $E\{v_k x_i^T\} = 0$, also $E\{w_k v_i^T\} = 0$

Notation:

$\hat{x}_k(-)$ <u>a priori</u> estimate of x_k conditioned on all measurements except the one at time t_k

$\hat{x}_k(+)$ <u>a posteriori</u> estimate of x_k conditioned on all measurements including the one at time t_k

Derivations of the Kalman filter will be done by using the orthogonality principle ("the optimal estimation error is orthogonal to data (measurement). This was done in the original Kalman's paper, <u>1960.</u>



$$\hat{x}_k(+) = (\quad)\, \hat{x}_k(-) + (\quad)\, z_k = K_k' \, \hat{x}_k(-) + \bar{K}_k z_k$$

updated estimate

to be determined from the orthogonality principle

Orthogonality principle $\Rightarrow$

$$\boxed{E\{(x_k - \hat{x}_k(+)) \cdot z_i^T\} = 0, \quad \forall i = 1, 2, \ldots, k-1,} \text{ (true also for } k\text{)}$$

Using the system and measurement equations and the estimate updated equation we have

$$E\{(\Phi_{k-1} x_{k-1} + w_{k-1} - K_k^1 \hat{x}_k(-) - \bar{K}_k z_k) \cdot z_i^T\} = 0$$

can be dropped due to "uncorrelated assumption"

$$E\{w_{k-1} (H_i x_i + v_i)^T\} = 0$$

By replacing z_k we get

$$E\{(\Phi_{k-1} x_{k-1} - K_k^1 \hat{x}_k(-) - \bar{K}_k H_k x_k - \bar{K}_k v_k) z_i^T\} = 0$$

$$= \Phi_{k-1} E\{x_{k-1} z_i^T\} - K_k^1 E\{\hat{x}_k(-) z_i^T\} - \bar{K}_k H_k E\{(\Phi_{k-1} x_{k-1} + w_{k-1}) z_i^T\}$$
$$- \bar{K}_k E\{v_k \cdot z_i^T\} = 0$$

$i = 1, 2, \ldots, k-1$

Note that $E\{v_k z_i^T\} = 0, \; i = 1, 2, \ldots, k-1$

$$= E\{v_k x_i^T\} H_k^T + E\{v_k v_i^T\} = 0 \quad i = 1, 2, \ldots, k-1$$

$$= \Phi_{k-1} E\{x_{k-1} z_i^T\} - K_k^1 E\{\hat{x}_k(-) z_i^T\} - \bar{K}_k H_k \Phi_{k-1} E\{x_{k-1} z_i^T\} = 0$$

$$= E\{x_k z_i^T\} - \bar{K}_k H_k E\{x_k z_i^T\} - K_k^1 E\{\hat{x}_k(-) z_i^T\} \pm E\{K_k^1 x_k z_i^T\} = 0$$

$$= E\{(x_k - \bar{K}_k H_k x_k - K_k^1 x_k) z_i^T\} - K_k^1 E\{\underbrace{(\hat{x}_k(-) - x_k)}_{\text{estimation error}} \underbrace{z_i^T}_{\text{data}}\} = 0$$

$\Rightarrow = 0$

$$= (I - \bar{K}_k H_k - K_k^1) E\{x_k z_i^T\} = 0$$

This is true for every x_k if

$$(I - \bar{K}_k H_k - K_k^1) = 0 \quad \Rightarrow \quad \boxed{K_k^1 = I - \bar{K}_k H_k} \qquad (1)$$

The orthogonality principle is also valid for $\underline{i=k}$

$$(*) \quad \boxed{E\{(x_k - \hat{x}_k(+))\, z_k^T\} = 0}$$

Introduce the errors as

$$\left.\begin{aligned} \tilde{x}_k(+) &= \hat{x}_k(+) - x_k \\ \tilde{x}_k(-) &= \hat{x}_k(-) - x_k \end{aligned}\right\} \text{ estimation errors after and before updates}$$

$$\tilde{z}_k = \hat{z}_k(-) - z_k = H_k \hat{x}_k(-) - z_k$$

Also, we have

$$E\{(x_k - \hat{x}_k(+))\, \hat{z}_k^T(-)\} = 0 \quad \text{why?} \quad \hat{z}_k(-) = H_k \hat{x}_k(-)$$

and

$$E\{(x_k - \hat{x}_k(+))\, \tilde{z}_k^T\} = 0$$

$\Rightarrow$

$$E\{(\Phi_{k-1} x_{k-1} + \cancel{w_{k-1}} - K_k^1 \hat{x}_k(-) - \bar{K}_k z_k)(H_k \hat{x}_k(-) - z_k)^T\} = 0$$

can be dropped since

$$E\{w_{k-1} z_k^T\} = 0 \qquad E\{w_{k-1} \hat{x}_k^T(-)\} = 0$$

$$E\{[\Phi_{k-1} x_{k-1} - \hat{x}_k(-) + \bar{K}_k H_k \hat{x}_k(-) - \bar{K}_k H_k x_k - \bar{K}_k v_k][H_k \hat{x}_k(-) - H_k x_k - v_k]^T\}$$

$$[H_k \tilde{x}_k(-) - v_k]^T\}$$

$$= E\{[(x_k - \hat{x}_k(-)) - \bar{K}_k H_k (x_k - \hat{x}_k(-)) - \bar{K}_k v_k][H_k \tilde{x}_k(-) - v_k]^T\}$$

$$= E\{[-\tilde{x}_k(-) + \bar{K}_k H_k \tilde{x}_k(-) - \bar{K}_k v_k][H_k \tilde{x}_k(-) - v_k]^T\}$$

Define $\quad P_k(-) = E\{\tilde{x}_k(-)\, \tilde{x}_k^T(-)\} =$ a priori error covariance

$\Rightarrow$

$$(I - \bar{K}_k H_k) P_k(-) H_k^T - \bar{K}_k R_k = 0$$

$$P_k(-) H_k^T = \bar{K}_k H_k P_k(-) H_k^T + \bar{K}_k R_k$$

$$\bar{K}_k (H_k P_k(-) H_k^T + R_k) = P_k(-) H_k^T$$

$$\boxed{\bar{K}_k = P_k(-) H_k^T (H_k P_k(-) H_k^T + R_k)^{-1}} \qquad (2)$$

Define

$$P_k(+) = E\{\tilde{x}_k(+)\,\tilde{x}_k^T(+)\} = \text{a posteriori covariance}$$

know that

$$\hat{x}_k(+) = K_k^1\,\hat{x}_k(-) + \bar{K}_k z_k = (I - \bar{K}_k H_k)\,\hat{x}_k(-) + \bar{K}_k z_k$$

$$\boxed{\hat{x}_k(+) = \hat{x}_k(-) + \bar{K}_k\,(z_k - H_k\hat{x}_k(-))}$$

$$\hat{x}_k(+) - x_k = -x_k + \bar{K}_k z_k - \bar{K}_k H_k \hat{x}_k(-) + \hat{x}_k(-)$$

$$= -x_k + \bar{K}_k H_k x_k + \bar{K}_k v_k - \bar{K}_k H_k \hat{x}_k(-) + \hat{x}_k(-)$$

$$\tilde{x}_k(+) = K_k H_k (x_k - \hat{x}_k(-)) + \bar{K}_k v_k + \underbrace{(\hat{x}_k(-) - x_k)}_{\tilde{x}_k(-)}$$

$$= (I - \bar{K}_k H_k)\,\tilde{x}_k(-) + \bar{K}_k v_k$$

so that

$$P_k(+) = E\{\tilde{x}_k(+)\,\tilde{x}_k^T(+)\} =$$

$$= E\{[(I-\bar{K}_k H_k)\tilde{x}_k(-) + \bar{K}_k v_k][(I-\bar{K}_k H_k)\tilde{x}_k(-) + \bar{K}_k v_k]^T$$

$$= (I - \bar{K}_k H_k)\,\underbrace{E\{\tilde{x}_k(-)\,\tilde{x}_k^T(-)\}}_{P_k(-)}\,(I - \bar{K}_k H_k)^T + \bar{K}_k \underbrace{E\{v_k v_k^T\}}_{R_k} \bar{K}_k$$

thus

$$\boxed{P_k(+) = (I - \bar{K}_k H_k)\,P_k(-)\,(I - \bar{K}_k H_k)^T + \bar{K}_k R_k \bar{K}_k^T} \qquad (3)$$

COVARIANCE UPDATE EQUATION (Joseph's form)

where

$$\boxed{\bar{K}_k = P_k(-)H_k^T\,(H_k P_k(-) H_k^T + R_k)^{-1}} \qquad (2)$$

Another form for (3)

$$P_k(+) = P_k(-) - \bar{K}_k H_k P_k(-) - P_k(-) H_k^T \bar{K}_k^T + \bar{K}_k H_k P_k(-) H_k^T \bar{K}_k^T + \bar{K}_k R_k \bar{K}_k^T$$

$$= (I - \bar{K}_k H_k)\,P_k(-) - P_k(-) H_k^T \bar{K}_k^T + \underbrace{\bar{K}_k\,(R_k + H_k P_k(-) H_k^T)}_{\overset{(2)}{=}\,P_k(-)H_k^T} \bar{K}_k^T$$

$$\boxed{P_k(+) = (I - \bar{K}_k H_k)\,P_k(-)} \qquad (4)$$

A priori covariance

$$P_k(-) = E\{\tilde{x}_k(-)\,\tilde{x}_k^T(-)\}$$

Note that $\boxed{\hat{x}_k(-) = \Phi_{k-1}\,\hat{x}_{k-1}(+)}$

the best that one can do at k assuming that the optimal estimate at $k-1$ is known.

Then

$$\hat{x}_k(-) - x_k = \Phi_{k-1}\hat{x}_{k-1}(+) - x_k = \Phi_{k-1}\hat{x}_{k-1}(+) - \Phi x_{k-1} - w_{k-1}$$

$$\tilde{x}_k(-) = \Phi_{k-1}(\underbrace{\hat{x}_{k-1}(+) - x_{k-1}}_{\tilde{x}_{k-1}(+)}) - w_{k-1}$$

$$\tilde{x}_k(-) = \Phi_{k-1}\,\tilde{x}_{k-1}(+) - w_{k-1}$$

$$E\{\tilde{x}_k(-)\,\tilde{x}_k^T(-)\} = \Phi_{k-1}\,E\{\tilde{x}_{k-1}(+)\,\tilde{x}_{k-1}^T(+)\}\,\Phi_{k-1}^T + Q_{k-1}$$

$$\boxed{P_k(-) = \Phi_{k-1}\,P_{k-1}(+)\,\Phi_{k-1}^T + Q_{k-1}}$$

A priori value for $P_k(-)$

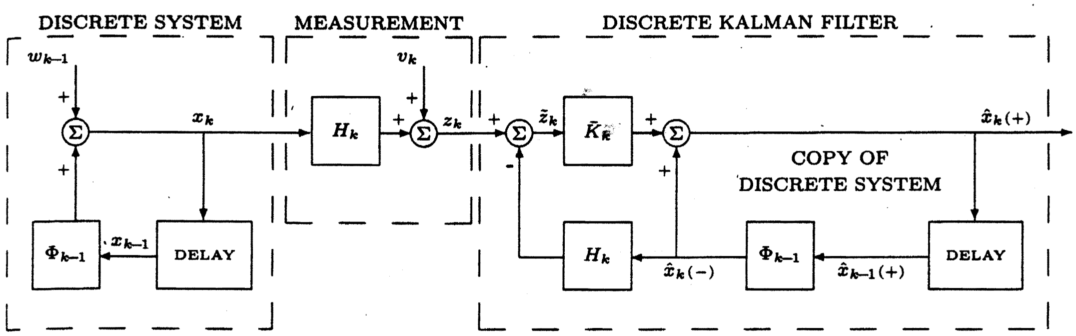


Figure 4.1 Block diagram of system, measurement model, and discrete-time Kalman filter.

4.2.1 Summary of Equations for the Discrete-Time Kalman Estimator

The equations derived in the preceding section are summarized in Table 4.3. In this formulation of the filter equations, G has been combined with the plant covariance by multiplying G_{k-1} and G_{k-1}^T, e.g.,

$$\begin{aligned} Q_{k-1} &= G_{k-1}\,\mathrm{E}(w_{k-1}w_{k-1}^T)G_{k-1}^T \\ &= G_{k-1}\bar{Q}_{k-1}G_{k-1}^T. \end{aligned}$$

TABLE 4.3 DISCRETE-TIME KALMAN FILTER EQUATIONS

System dynamic model:		
x_k	$=$	$\Phi_{k-1}x_{k-1} + w_{k-1}$
w_k	$\sim$	$\mathcal{N}(0, Q_k)$
Measurement model:		
z_k	$=$	$H_k x_k + v_k$
v_k	$\sim$	$\mathcal{N}(0, R_k)$
Initial conditions:		
$\mathrm{E}\langle x_0\rangle$	$=$	$\hat{x}_0$
$\mathrm{E}\langle \tilde{x}_0\tilde{x}_0^T\rangle$	$=$	P_0
Independence assumption:		
$\mathrm{E}\langle w_k v_j^T\rangle$	$=$	0 for all $k \neq j$
State estimate extrapolation (Equation 4.25):		
$\hat{x}_k(-)$	$=$	$\Phi_{k-1}\hat{x}_{k-1}(+)$ prediction
Error covariance extrapolation (Equation 4.26):		
$P_k(-)$	$=$	$\Phi_{k-1}P_{k-1}(+)\Phi_{k-1}^T + Q_{k-1}$
State estimate observational update (Equation 4.21):		
$\hat{x}_k(+)$	$=$	$\hat{x}_k(-) + \bar{K}_k\left[z_k - H_k\hat{x}_k(-)\right]$
Error covariance update (Equation 4.24):		
$P_k(+)$	$=$	$\left[I - \bar{K}_k H_k\right]P_k(-)$
Kalman gain matrix (Equation 4.19):		
$\bar{K}_k$	$=$	$P_k(-)H_k^T\left[H_k P_k(-)H_k^T + R_k\right]^{-1}$

The relation of the filter to the system is illustrated in the block diagram of Figure 4.1. The basic steps of the computational procedure for the discrete-time Kalman estimator are as follows:

1. Compute $P_k(-)$ using $P_{k-1}(+)$, Φ_{k-1}, and Q_{k-1}.
2. Compute $\bar{K}_k$ using $P_k(-)$ (computed in step 1), H_k, and R_k.
3. Compute $P_k(+)$ using $\bar{K}_k$ (computed in step 2) and $P_k(-)$ (from step 1).
4. Compute successive values of $\hat{x}_k(+)$ recursively, using the computed values of $\bar{K}_k$ (from step 3), the given initial estimate $\hat{x}_0$, and the input data z_k.

Step 4 of the Kalman filter implementation [computation of $\hat{x}_k(+)$] can be implemented only for state vector propagation where simulator or real data sets are available. An example of this is given in Section 4.12.