SUMMARY OF DEFINITIONS

Feb. 7, 97 (3)

Correlation matrix:

$$E\{x(t_1)\, x^T(t_2)\}$$

Covariance matrix:

$$E\{(x(t_1) - E(x(t_1)))(x(t_2) - E(x(t_2)))^T\}$$

Cross-Covariance matrix:

$$E\{(x(t_1) - E(x(t_1)))(y(t_2) - E(y(t_2)))^T\}$$

<u>Uncorrelated</u> stochastic processes:

<u>Cross-covariance = 0</u>

<u>Orthogonal</u> stochastic processes

<u>Correlation matrix = 0</u> $= E\{x(t_1)\, y^T(t_2)\}$

<u>Autocorrelation</u>:

$$E\{x(t)\, x(t+\tau)\} = \psi_x(\tau)$$

<u>Power Spectral Density</u> = Fourier transform of $\psi_x(\tau)$

(SPECTRUM

$$\Psi_x(\omega) = \mathcal{F}\{\psi_x(\tau)\} = \int_{-\infty}^{+\infty} \psi_x(\tau)\, e^{-j\omega\tau}\, d\tau$$

<u>White Noise</u> Stochastic Process: $w(t)$ with $E\{w(t)\} =$

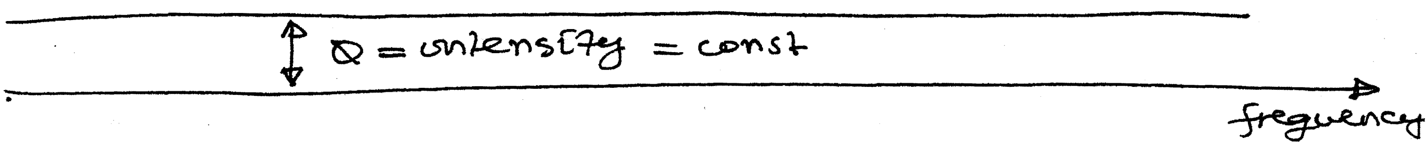
and $E\{w(t_1)\, w^T(t_2)\} = Q(t_1, t_2)\, \delta(t_1 - t_2)$

<u>Wide-sense stationary</u>: stochastic process

$$E\{x(t)\} = \text{const}$$

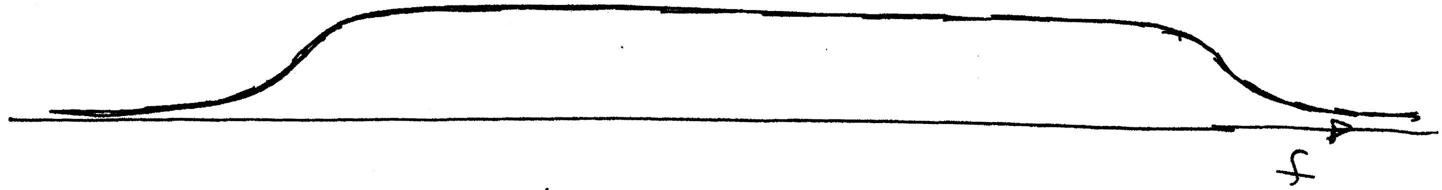
$$E\{x(t_1)\, x^T(t_2)\} = Q(t_2 - t_1) = Q(\tau)$$

White Noise Spectrum = flat spectrum = const $\forall f$

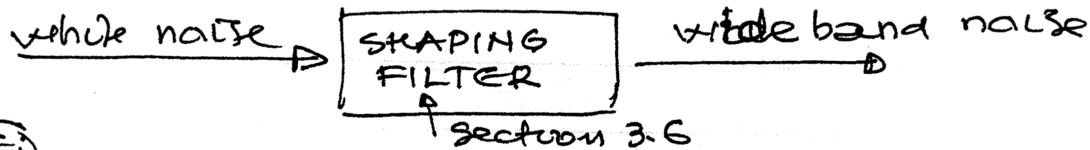


White Light, wind are approximated very well by white noise

<u>Wide band noise</u> (exponentially correlated noise)

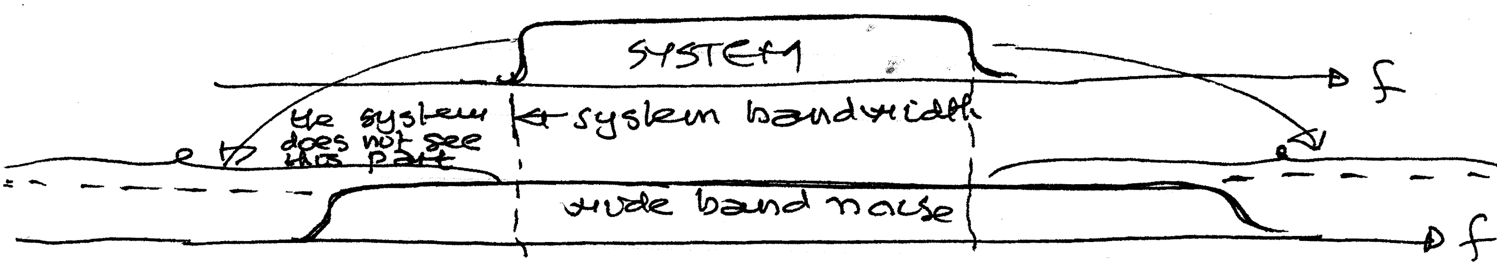


It can be modeled as an output of a system driven by white noise

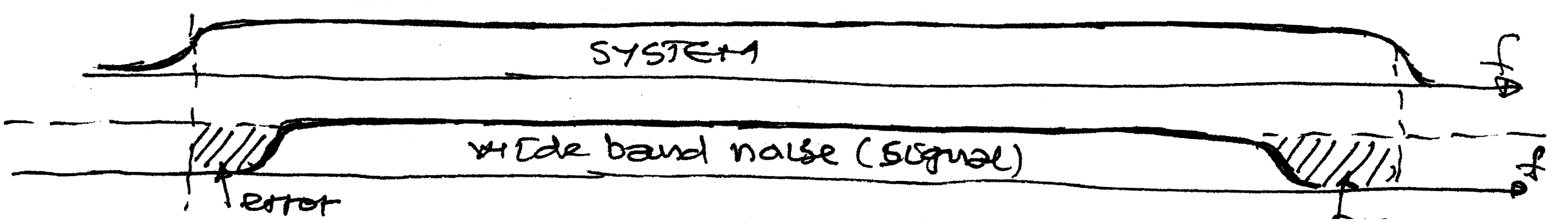


NOTE:

① If the system frequency bandwidth is not very large than <u>the wideband noise can be directly approximated by the white noise process</u>

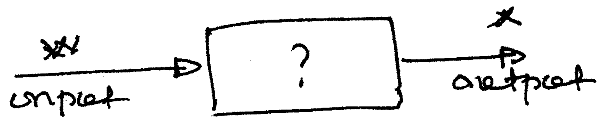


② If the system frequency bandwidth is large we get the approximation error due to the above approximation



(Example 3.5)

$$\psi_x(\tau) = \sigma^2 e^{-\alpha|\tau|} \longleftrightarrow \Psi_x(\omega) = \frac{2\sigma^2\alpha}{\omega^2+\alpha^2}$$



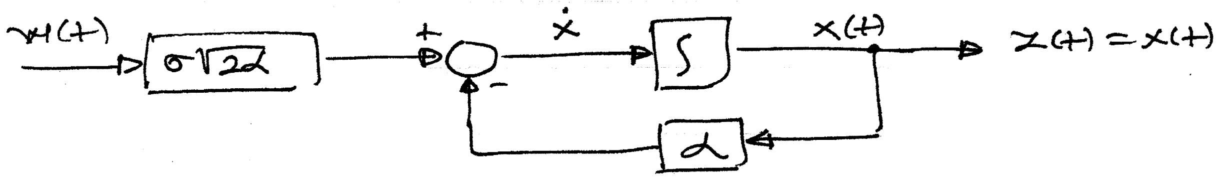
Know that the output spectrum is given by

$$\Psi_x(\omega) = \underbrace{|H(j\omega)|^2 \Psi_w(\omega)}_{\text{white noise (take } Q=1)} \qquad |H(j\omega)|^2 = H(j\omega)H^*(j\omega) = H(j\omega)H(-j\omega)$$

$$H(j\omega)\cdot H(-j\omega) = \frac{2\sigma^2\alpha}{\omega^2+\alpha^2} = \frac{\sqrt{2\alpha}\,\sigma}{\alpha+j\omega}\cdot\frac{\sqrt{2\alpha}\,\sigma}{\alpha-j\omega}$$

$$\Rightarrow H(j\omega) = \frac{\sqrt{2\alpha}\,\sigma}{\alpha+j\omega} = \frac{X(j\omega)}{W(j\omega)}$$

or $\dfrac{X(s)}{W(s)} = \dfrac{\sqrt{2\alpha}\,\sigma}{\alpha+s} \Rightarrow \dot{x}(t) = -\alpha x(t) + \sqrt{2\alpha}\,\sigma\, w(t)$

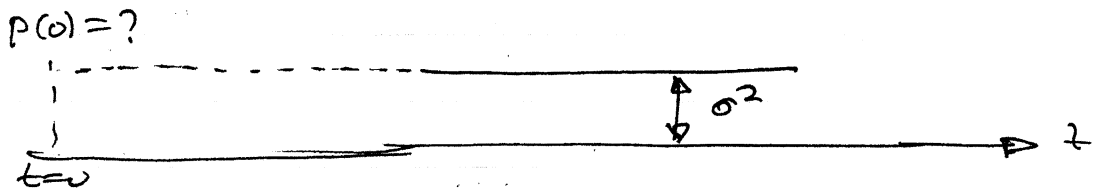


Note that

$$\dot{P} = \mathrm{Var}(x(t)) = -\alpha P(t) - \alpha P(t) + \sqrt{2\alpha}\,\sigma\cdot\sqrt{2\alpha}\,\sigma \quad \text{(Lyapunov eq.)}$$

which at steady state produces ($\dot{P}=0$)

$$0 = -2\alpha P + 2\alpha\sigma^2 \Rightarrow P = \sigma^2$$

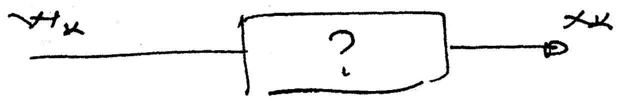


by choosing $P(0) = \sigma^2$ we eliminate transient so that $P(t) = \sigma^2$ for $\forall t$.

$$\Rightarrow \mathrm{Var}(x(t)) = \sigma^2$$

(Example 3.6) Discrete-Time Shaping Filter

$$\psi_x(k_2-k_1) = \sigma^2 e^{-\alpha|k_2-k_1|} \qquad \sigma^2 = \text{variance of } x$$



Let

$$x_k = \phi x_{k-1} + G w_{k-1}, \quad \phi, G = ?$$
$$z_k = x_k$$

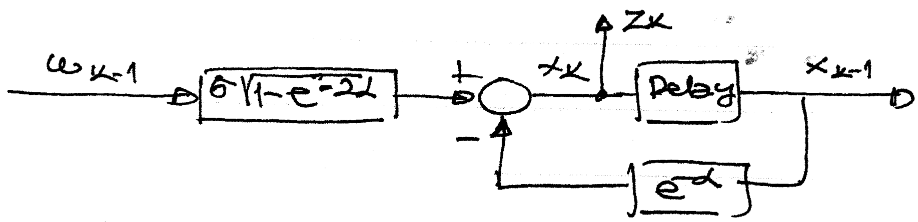
$$E\{x_k x_{k-1} = \phi x_{k-1} x_{k-1} + G w_{k-1} x_{k-1}\}$$
$$\sigma^2 e^{-\alpha} = \Phi \sigma^2 + 0$$ (it is assumed that w_{k-1} and x_{k-1} are uncorrelated)

$$\Rightarrow \quad \Phi = e^{-\alpha}$$

$$E\{x_k x_k = \phi^2 x_{k-1} x_{k-1} + G^2 w_{k-1} w_{k-1}\}$$
$$\sigma^2 = \sigma^2 \Phi^2 + G^2 \quad \Rightarrow \quad G = \sigma\sqrt{1-e^{-2\alpha}}$$

So that the required linear system is given by

$$x_k = e^{-\alpha} x_{k-1} + \sigma\sqrt{1-e^{-2\alpha}}\, w_{k-1}, \qquad E\{w_k\} = 0$$
$$z_k = x_k \qquad E\{w_{k_1} w_{k_2}\} = \Delta(k_2-k_1)$$



3.7 Covariance Propagation Equations

3.7.1 Continuous-time

The variance matrix for state variables = covariance matrix for error

$$P(t) = E\{\underbrace{(x(t) - E(x(t)))}_{e(t)}(x(t) - E(x(t)))^T\} = E\{e(t)e^T(t)\}$$

where

$$\dot{x} = F(t)x + G(t)w(t), \quad \begin{cases} E\{x(t_0)\} = \bar{x}_0, & Var\{x(t_0)\} = P(t_0) \\ E(w(t)) = 0, & \underline{E\{x(t_0)w(t)\} = 0} \\ E(w(t_1)w(t_2)) = Q(t_2, t_1)\delta(t_2 - t_1) \end{cases}$$

Know that

$$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, \tau)G(\tau)w(\tau)d\tau$$

$$\Rightarrow$$

$$E\{x(t)\} = \Phi(t, t_0)E(x(t_0)) + \int_{t_0}^{t} \Phi(t, \tau)G(\tau)\cancel{E\{w(\tau)\}}^{\,0}d\tau$$

MEAN $\boxed{E\{x(t)\} = \Phi(t, t_0)E\{x(t_0)\}}$, note $\dot{\Phi}(t, t_0) = F(t)\Phi(t, t_0)$

$\Rightarrow \Phi(t, t_0)$ deterministic $\quad \Phi(t_0, t_0) = I$

The variance matrix is now given by

$$P(t) = E\{(x(t) - E(x(t)))\,(x(t) - E(x(t)))^T\}$$

$$= E\{\Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, \tau)G(\tau)w(\tau)d\tau - \Phi(t, t_0)E(x(t_0))\} \times$$

$$\times \{\Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, \tau)G(\tau)w(\tau)d\tau - \Phi(t, t_0)E(x(t_0))\}^T$$

$$= E\{\Phi(t, t_0)\overbrace{\cancel{x(t_0)x^T(t_0)}}^{[x(t_0) - E(x(t_0))][x(t_0) - E(x(t_0))]^T}\Phi^T(t, t_0)\} + (\quad)E\{x(t_0)w^T(\tau)\}(\quad)$$

$$+ E\{\int_{t_0}^{t}\int_{t_0}^{t} \Phi(t, \tau_1)G(\tau_1)w(\tau_1)w^T(\tau_2)G^T(\tau_2)\Phi^T(t, \tau_2)d\tau_1 d\tau_2$$

$$+ (\quad)E\{w(\tau)x^T(t_0)\}(\quad)$$

$$= \Phi(t, t_0)Var\{x(t_0)\}\Phi^T(t, t_0) + 0 + 0$$

$$+ \int_{t_0}^{t}\int_{t_0}^{t} \Phi(t, \tau_1)G(\tau_1)\underbrace{E\{w(\tau_1)w^T(\tau_2)\}}_{= Q\delta(\tau_1 - \tau_2)}G^T(\tau_2)\Phi^T(t, \tau_2)d\tau_1 d\tau_2$$

$$P(t) = \Phi(t,t_0)\,P(t_0)\,\Phi^T(t,t_0) + \int_{t_0}^{t} \Phi(t,\tau)G(\tau)\,Q\,G^T(\tau)\,\Phi^T(t,\tau)\,d\tau$$

Take the derivative and use the formula

$$\frac{d}{dt}\int_{L(t)}^{U(t)} f(t,\sigma)\,d\sigma = \int_{L(t)}^{U(t)} \frac{\partial}{\partial t} f(t,\sigma)\,d\sigma + f(U(t),t)\frac{\partial U(t)}{\partial t} - f(L(t),t)\frac{\partial L(t)}{\partial t}$$

$$\dot{P}(t) = \dot{\Phi}(t,t_0)P(t_0)\Phi^T(t,t_0) + \Phi(t,t_0)P(t_0)\dot{\Phi}^T(t,t_0)$$
$$+ \Phi(t,t)G(t)Q\,G^T(t)\Phi^T(t,t)\cdot 1 - (\quad)\cdot 0$$
$$+ \int_{t_0}^{t}\left[\dot{\Phi}(t,\tau)G(\tau)Q\,G^T(\tau)\Phi^T(t,\tau) + \Phi(t,\tau)G(\tau)Q\,G^T(\tau)\dot{\Phi}^T(t,\tau)\right]d\tau$$

$$\dot{\Phi}(t,t_0) = F(t)\,\Phi(t,t_0)\quad,\quad \Phi(t_0,t_0) = I$$

$$\dot{P}(t) = F(t)\Phi(t,t_0)P(t_0)\Phi^T(t,t_0) + \Phi(t,t_0)P(t_0)\Phi^T(t,t_0)F^T(t)$$
$$+ I\,G(t)Q\,G^T(t)\cdot I + F(t)\int_{t_0}^{t}\left[\Phi(t,\tau)G(\tau)Q\,G^T(\tau)\Phi^T(t,\tau)\right]d\tau$$
$$+ \int_{t_0}^{t}(\Phi(t,\tau)G(\tau)Q\,G^T(\tau)\Phi^T(t,\tau)d\tau\cdot F^T(t))$$

$\Rightarrow$

$$\boxed{\dot{P}(t) = F(t)P(t) + P(t)F^T(t) + G(t)QG^T(t)\quad,\quad P(t_0) = P_0 = \text{given}}$$

at steady state $\dot{P}(t) = 0$ (assuming that $F(t) = F = \text{const}$, $G(t) = G = \text{const}$, and F stable)

$\Rightarrow$

$$\boxed{0 = FP(\infty) + P(\infty)F^T + GQG^T}$$

Algebraic Lyapunov equation in continuous-time

Note that when F is stable

$\Rightarrow$ a unique solution for $P(\infty)$ exists.

"LYAPUNOV EQUATION IN SYSTEM STABILITY and CONTROL"
by Z. GAJIC and M. QURESHI, Academic Press, 1995

(3.7.2) Discrete-Time

$$P_k = E\{[x_k - E(x_k)][x_k - E(x_k)]^T\}$$

$$x_k = \Phi_{k-1} x_{k-1} + w_{k-1}, \quad \begin{cases} E(x_0) = \bar{x}_0, \ Var(x_0) = P_0 \\ E(w_{k-1}) = 0, \ E\{w_k w_k^T\} = Q_k \\ E\{w_k x_k\} = 0 \quad \forall k \end{cases}$$

$$P_k = E\{[\Phi_{k-1} x_{k-1} + w_{k-1} - E(x_k)][\Phi_{k-1} x_{k-1} + w_{k-1} - E(x_k)]^T\}$$

note that $\quad E\{x_k\} = \Phi_{k-1} E\{x_{k-1}\} + 0$

$$P_k = E\{[\Phi_{k-1}(x_{k-1} - E(x_{k-1})) + w_{k-1}][w_{k-1}^T + (x_{k-1} - E(x_{k-1}))^T \Phi_{k-1}^T]\}$$

$$= \Phi_{k-1} E\{(x_{k-1} - E(x_{k-1}))(x_{k-1} - E(x_{k-1}))^T\} \Phi_{k-1}^T + E\{w_{k-1} w_{k-1}^T\}$$

$$+ \Phi_{k-1} \underbrace{E\{(x_{k-1} - E(x_{k-1})) w_{k-1}^T\}}_{0} + \underbrace{E\{ \qquad \}}_{0} \Phi_{k-1}^T$$

$\Rightarrow$

$$\boxed{P_k = \Phi_{k-1} P_{k-1} \Phi_{k-1}^T + Q_{k-1}}$$

with $P_0 = Var(x_0)$ = initial condition

The variance equation = Difference Lyapunov Equation

Note that

$$E\{(x_{k-1} - E(x_{k-1})) w_{k-1}^T\}$$

$$= \underbrace{E\{x_{k-1} w_{k-1}^T\}}_{0} - E(x_{k-1}) \underbrace{E(w_{k-1}^T)}_{0} = 0$$

At steady state (k large, $P_k \to P_{k-1}$)

$$P_k = P_{k-1} = P_\infty$$

$$\boxed{P_\infty = \Phi P_\infty \Phi^T + Q}$$

Discrete-time Algebraic Lyapunov equation

Note that the steady state can be reached for constant matrices Φ and Q. In addition, the matrix Φ must be stable.

$P_\infty = \Phi P_\infty \Phi^T + Q$

Note that the stability of Φ (All real part of $\lambda(\Phi)$ are inside of the unit circle and those on the unit circle ($\lambda(\Phi) \in$ unit circle) are distinct (with multiplicity equal to one)) implies the existence of a unique solution for P_∞.

3.39 Find the covariance matrix $P(t)$ and its steady-state value $P(\infty)$ for the following continuous systems:

$$\text{(a)} \qquad \dot{x} = \begin{bmatrix} 0 & 0 \\ -1 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w(t)$$

$$P(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{(b)} \qquad \dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 5 \\ 1 \end{bmatrix} w(t)$$

$$P(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

where $w \in \mathcal{N}(0,1)$ and white.

SOLUTION TO EXERCISE 3.39 **(a)**

$$\begin{aligned} \dot{P} &= FP + PF^T + GQG^T \\ &= \begin{bmatrix} 0 & 0 \\ -1 & -2 \end{bmatrix} P + P \begin{bmatrix} 0 & -1 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 - p_{11} - 2p_{12} \\ 1 - p_{11} - 2p_{12} & 1 - 2p_{12} - 4p_{22} \end{bmatrix}, \end{aligned}$$

which has no steady-state solution for $\dot{P} = 0$.

(b)

$$\begin{aligned} \dot{P} &= FP + PF^T + GQG^T \\ &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} P + P \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 25 & 5 \\ 5 & 1 \end{bmatrix} \end{aligned}$$

—yielding

$$p_{11} = \frac{25}{2}$$

3.18 Find the state space models for longitudinal, vertical and lateral turbulence for the following PSD of the "Dryden" turbulence model.

$$\Psi(\omega) = \sigma^2 \left(\frac{2L}{\pi V}\right) \left(\frac{1}{1 + \left(\frac{L\omega}{V}\right)^2}\right)$$

$$\begin{aligned} \Psi(\omega) &= \text{Frequency in Rad/Sec} \\ \sigma &= \text{RMS turbulence intensity} \\ L &= \text{Scale length in feet} \\ V &= \text{Airplane velocity in feet/sec. (290ft/sec)} \end{aligned}$$

(a) Longitudinal turbulence

$$L = 600'$$

$$\sigma_u = .15 \quad \text{mean head wind or tail wind (knots)}$$

(b) Vertical turbulence

$$L = 300'$$

$$\sigma_w = 1.5 \ \text{knots}$$

(c) Lateral turbulence

$$L = 600'$$

$$\sigma_v = .15 \ \text{mean crosswind (knots)}$$

Solution to Exercise 3.18 (a)

$$\begin{aligned} \psi(\omega) &= \left[\sigma \left(\frac{2L}{\pi v}\right)^{\frac{1}{2}} \frac{1}{1 + j\omega\frac{L}{v}}\right] \left[\sigma \left(\frac{2L}{\pi v}\right)^{\frac{1}{2}} \frac{1}{1 - j\omega\frac{L}{v}}\right] \\ &= H(j\omega)H(-j\omega), \end{aligned}$$

where $H(j\omega)$ is stable. For $s = j\omega$,

$$\begin{aligned} \left(1 + s\frac{L}{v}\right) x(x) &= \sigma \left(\frac{2L}{\pi v}\right)^{\frac{1}{2}} w(s) \text{ , or} \\ \dot{x}(t) &= -\frac{v}{L} x(t) + \sigma \left(\frac{2L}{\pi v}\right)^{\frac{1}{2}} w(t) \\ V &= 290 \text{ ft/sec} \\ L &= 600 \text{ ft} \\ \sigma &= 0.15 \\ \dot{x}(t) &= -0.483x(t) + 0.083w(t) \\ z(t) &= x(t) \end{aligned}$$

(b)

$$\dot{x}(t) = -0.967x(t) + 0.118w(t)$$

(c)

$$\dot{x}(t) = -0.483x(t) + 0.083w(t)$$

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$$\begin{aligned} p_{12} &= \frac{5}{2} \\ p_{22} &= \frac{1}{2} \end{aligned}$$

for the steady-state solution to $\dot{P} = 0$.

3.40 Find the covariance matrix P_k and its steady-state value P_∞ for the following discrete system

$$\begin{aligned} x_{k+1} &= \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w_k \\ P_0 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

where $w_k \in \mathcal{N}(0,1)$ and white.

SOLUTION TO EXERCISE 3.40 Because $w_k \in \mathcal{N}(0,1)$, $Q = 1$ and the matrix product

$$GQG^T = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

The matrix equation

$$P_{k+1} = \Phi P_k \Phi^T + GQG^T$$

has the specific form

$$\begin{aligned} \begin{bmatrix} \{P_{k+1}\}_{11} & \{P_{k+1}\}_{12} \\ \{P_{k+1}\}_{12} & \{P_{k+1}\}_{22} \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} \{P_k\}_{11} & \{P_k\}_{12} \\ \{P_k\}_{12} & \{P_k\}_{22} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 + \{P_k\}_{22} & 1 - \{P_k\}_{12} + 2\{P_k\}_{22} \\ 1 - \{P_k\}_{12} + 2\{P_k\}_{22} & 1 + \{P_k\}_{11} - 4\{P_k\}_{12} + 4\{P_k\}_{22} \end{bmatrix}. \end{aligned}$$

This is a *linear* equation in the three unique elements of P, and it can be expressed in vector form as

$$\begin{bmatrix} \{P_{k+1}\}_{11} \\ \{P_{k+1}\}_{12} \\ \{P_{k+1}\}_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -4 & 4 \end{bmatrix} \begin{bmatrix} \{P_k\}_{11} \\ \{P_k\}_{12} \\ \{P_k\}_{22} \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Because

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -4 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

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the general solution can be written in closed form as

$$
\begin{aligned}
\begin{bmatrix} \{P_k\}_{11} \\ \{P_k\}_{12} \\ \{P_k\}_{22} \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -4 & 4 \end{bmatrix}^k \begin{bmatrix} \{P_0\}_{11} \\ \{P_0\}_{12} \\ \{P_0\}_{22} \end{bmatrix} + k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -4 & 4 \end{bmatrix}^k \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.
\end{aligned}
$$

Note that $\{P_k\}_{ij} \to \infty$ as $k \to \infty$. That is, P_∞ is unbounded.

3.41 Find the steady-state covariance for the state space model given in Example 3.4.

SOLUTION TO EXERCISE 3.41 This exercise demonstrates how to transform the steady-state equation for the state covariance into a system of linear equations.

The dynamic equation for the state covariance is

$$
\begin{aligned}
\dot{P}(t) &= FP(t) + P(t)F^T + GQG^T \\
P(t) &= \mathrm{E}\left\langle p(t)p^T(t)\right\rangle \\
F &= \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \\
G &= \begin{bmatrix} a \\ b - 2a\zeta\omega_n \end{bmatrix} \\
Q &= \mathrm{E}\left\langle w^2(t)\right\rangle.
\end{aligned}
$$

Its steady-state form is

$$
\begin{aligned}
P(\infty) &= \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \\
0 &= \dot{P}(\infty) \\
&= FP(\infty) + P(\infty)F^T + GQG^T \\
&= \begin{bmatrix} Qa^2 + 2p_{12} & Qa\left(b - 2a\omega_n\zeta\right) - \omega_n^2 p_{11} + p_{22} - 2\omega_n p_{12}\zeta \\ Qa\left(b - 2a\omega_n\zeta\right) - \omega_n^2 p_{11} + p_{22} - 2\omega_n p_{12}\zeta & Q\left(b - 2Qa\omega_n\zeta\right)^2 - 2\omega_n^2 p_{12} - 4\omega_n p_{22}\zeta \end{bmatrix}.
\end{aligned}
$$

This linear matrix equation is equivalent to the three scalar linear equations

$$
\begin{aligned}
2p_{12} &= -Qa^2 \\
\omega_n^2 p_{11} + 2\omega_n\zeta p_{12} - p_{22} &= Qa\left(b - 2a\omega_n\zeta\right) \\
2\omega_n^2 p_{12} + 4\omega_n\zeta p_{22} &= Q\left(b - 2a\omega_n\zeta\right)^2.
\end{aligned}
$$

This system of 3 linear equations in 3 unknowns has the solution

$$\begin{aligned}
p_{11} &= \frac{Q\left(b^2 + a^2\omega_n^2\right)}{4\omega_n^3\zeta} \\
p_{12} &= -Qa^2/2 \\
p_{22} &= \frac{Q\left[a^2\omega_n^2 + (b - 2a\omega_n\zeta)^2\right]}{4\omega_n\zeta}.
\end{aligned}$$

3.42 Show that the continuous-time steady-state algebraic equation

$$0 = FP(\infty) + P(\infty)F^T + GQG^T$$

has no nonnegative solution for the scalar case with $F = Q = G = 1$. . (See Equation 3.110.)

SOLUTION TO EXERCISE 3.42 The algebraic equation for $F = Q = G = 1$ is

$$\begin{aligned}
0 &= P(\infty) + P(\infty) + 1 \\
P(\infty) &= -1/2,
\end{aligned}$$

which is negative. That is, the only solution is negative, and there is no non-negative solution.

3.43 Show that the discrete time steady-state algebraic equation

$$P_\infty = \Phi \overline{P_\infty} \Phi^T + Q$$

has no solution for the scalar case with $\Phi = Q = 1$. . (See Equation 3.112.)

SOLUTION TO EXERCISE 3.43 The algebraic equation for $\Phi = Q = 1$ is

$$P_\infty = P_\infty + 1,$$

which is equivalent to

$$0 = 1,$$

independent of P_∞. Clearly, then, there is not value of Φ that can make $0 = 1$.