10-3. Linear systems (Papoulis) = Sec. 3.5 Grewal & Andrews
pages 344-345 pages 73-74

We are given a linear system with impulse response a certain function $h(t)$. The Fourier transform $H(j\omega)$ of $h(t)$ is known as *system* function†

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\,dt \tag{10-28}$$

We now apply to the input of our system a process $\mathbf{x}(t)$ (Fig. 10-4). As is well known, the resulting output $\mathbf{y}(t)$ is given by

$$\mathbf{y}(t) = \int_{-\infty}^{\infty} \mathbf{x}(t-\alpha)h(\alpha)\,d\alpha = \int_{-\infty}^{\infty} \mathbf{x}(\alpha)h(t-\alpha)\,d\alpha \tag{10-29}$$

† We used the notation $H(j\omega)$ to conform with the usual convention of reserving $H(p)$ for the Laplace transform of $h(t)$. For theoretical purposes [see (10-40)] we shall not exclude the possibility that $h(t)$ might be complex.

In physical systems, the impulse response $h(t)$ is real and it equals zero for negative t (causality). In this case, (10-29) takes the form

$$\mathbf{y}(t) = \int_{0}^{\infty} \mathbf{x}(t-\alpha)h(\alpha)\,d\alpha = \int_{-\infty}^{t} \mathbf{x}(\alpha)h(t-\alpha)\,d\alpha \tag{10-30}$$

However, in our analysis it will not be necessary to make this assumption.

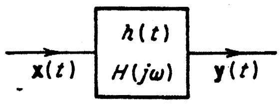

Fig. 10-4

The following discussion is a concrete version of the results presented in Sec. 9-5 in terms of linear operators.

Mean and autocorrelation. We now assume that the input $\mathbf{x}(t)$ is stationary. Reasoning as in (9-97), we conclude that

$$E\{\mathbf{y}(t)\} = \int_{-\infty}^{\infty} E\{\mathbf{x}(t-\alpha)\}h(\alpha)\,d\alpha$$

Thus the expected value of $\mathbf{y}(t)$ is constant and is given by

$$\boxed{\eta_y = \eta_x \int_{-\infty}^{\infty} h(\alpha)\,d\alpha = H(0)\eta_x} \tag{10-31}$$

To determine the autocorrelation of the output $\mathbf{y}(t)$, we shall first determine the cross-correlation between $\mathbf{x}(t)$ and $\mathbf{y}(t)$. Multiplying both sides of (10-29) by $\mathbf{x}^*(t-\tau)$, we have

$$\mathbf{y}(t)\mathbf{x}^*(t-\tau) = \int_{-\infty}^{\infty} \mathbf{x}(t-\alpha)\mathbf{x}^*(t-\tau)h(\alpha)\,d\alpha \tag{10-32}$$

But

$$E\{\mathbf{x}(t-\alpha)\mathbf{x}^*(t-\tau)\} = R_{xx}[(t-\alpha)-(t-\tau)] = R_{xx}(\tau-\alpha)$$

Hence, taking expected values of both sides of (10-32), we obtain

$$R_{yx}(\tau) = E\{\mathbf{y}(t)\mathbf{x}^*(t-\tau)\} = \int_{-\infty}^{\infty} R_{xx}(\tau-\alpha)h(\alpha)\,d\alpha$$

The above integral is obviously independent of t, and it equals the convolution of $R_{xx}(\tau)$ with $h(\tau)$. Hence the left-hand side is also independent of t, and since it equals the cross-correlation of $\mathbf{y}(t)$ and $\mathbf{x}(t)$, we conclude that

$$\boxed{R_{yx}(\tau) = R_{xx}(\tau) * h(\tau)} \tag{10-33}$$

Multiplying the conjugates of both sides of (10-29) by $\mathbf{y}(t+\tau)$, we also have

$$\mathbf{y}(t+\tau)\mathbf{y}^*(t) = \int_{-\infty}^{\infty} \mathbf{y}(t+\tau)\mathbf{x}^*(t-\alpha)h^*(\alpha)\,d\alpha$$

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Hence

$$\boxed{R_{yy}(\tau) = \int_{-\infty}^{\infty} R_{yx}(\tau + \alpha)h^*(\alpha)\, d\alpha = R_{yx}(\tau) * h^*(-\tau)} \quad (10\text{-}34)$$

The last equality resulted with $\alpha = -\beta$. Reasoning as above, we can similarly show that

$$R_{xy}(\tau) = R_{xx}(\tau) * h^*(-\tau) \qquad R_{yy}(\tau) = R_{xy}(\tau) * h(\tau) \quad (10\text{-}35)$$

Hence

$$\boxed{R_{yy}(\tau) = R_{xx}(\tau) * h^*(-\tau) * h(\tau)} \quad (10\text{-}36)$$

These relationships can be given a system interpretation: Applying $R_{xx}(\tau)$ to a system with impulse response $h^*(-\tau)$, we obtain as output $R_{xy}(\tau)$ (Fig. 10-5). With $R_{xy}(\tau)$ as input to the system $h(\tau)$, the output is $R_{yy}(\tau)$.

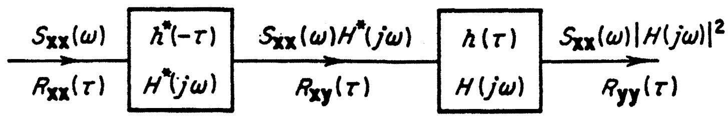


Fig. 10-5

Comments

1. If $\mathbf{x}(t)$ is white noise, i.e., if $R_{xx}(\tau) = \delta(\tau)$, and $h(t) = 0$ for $t < 0$ (real causal system), then

$$R_{xy}(\tau) = h(-\tau) = 0 \qquad \text{for } \tau > 0$$

In other words, $\mathbf{y}(t)$ is orthogonal to $\mathbf{x}(t + \tau)$ for $\tau > 0$.

2. Using white noise as input, one can measure experimentally the impulse response $h(t)$ of a system by a time average. This is done as follows: Since $R_{xy}(\tau) = h(-\tau)$, it suffices to measure the cross-correlation between the input $\mathbf{x}(t)$ and the resulting output $\mathbf{y}(t)$ for various negative values of τ. If $R_{xy}(\tau)$ is ergodic (see Prob. 9-20), then for sufficiently large T

$$h(-\tau) = R_{xy}(\tau) \simeq \frac{1}{T}\int_0^T \mathbf{x}(t + \tau)\mathbf{y}(t)\, dt$$

where $\mathbf{x}(t)$ is a single function of the input process, and $\mathbf{y}(t)$ is the resulting response. The last integral can be evaluated with a correlator (multiplier and integrator). If $\mathbf{x}(t)$ is not white noise, then $h(\tau)$ can be found by solving the integral equation

$$R_{xy}(\tau) = \int_{-\infty}^{0} R_{xx}(\tau - \alpha)h(-\alpha)\, d\alpha$$

Its solution is simple, using transforms: $H^*(j\omega) = S_{xy}(\omega)/S_{xx}(\omega)$ [see

(10-37)]. However, computationally, it might be best to use other methods.†

Stationarity of the output. From the preceding discussion follows that if $\mathbf{x}(t)$ is wide-sense stationary, then the mean of $\mathbf{y}(t)$ is constant and its autocorrelation a function only of τ. Hence $\mathbf{y}(t)$ is also wide-sense stationary.

The same is true for strict-sense stationarity: From

$$\mathbf{y}(t + \varepsilon) = \int_{-\infty}^{\infty} \mathbf{x}(t + \varepsilon - \alpha)h(\alpha)\, d\alpha$$

we conclude that if the processes $\mathbf{x}(t)$ and $\mathbf{x}(t + \varepsilon)$ have the same statistics, then the same is true for $\mathbf{y}(t)$ and $\mathbf{y}(t + \varepsilon)$.

The above conclusions are not correct if the input to the system is applied at $t = 0$. In this case, if $h(t)$ is absolutely integrable (stable system), then $\mathbf{y}(t)$ is asymptotically stationary.

Power Spectrum

Since‡ the Fourier transform of $h^*(-t)$ equals $H^*(j\omega)$, we conclude from (10-35) and the convolution theorem (page 159) that

$$S_{xy}(\omega) = S_{xx}(\omega)H^*(j\omega) \qquad S_{yy}(\omega) = S_{xy}(\omega)H(j\omega) \tag{10-37}$$

Combining, we obtain:

Fundamental theorem. The power spectrum $S_{yy}(\omega)$ of the output of a linear system with system function $H(j\omega)$ is given by

$$\boxed{S_{yy}(\omega) = S_{xx}(\omega)|H(j\omega)|^2} \tag{10-38}$$

where $S_{xx}(\omega)$ is the power spectrum of the input. $\Leftarrow \mathcal{F}\{(10\text{-}36)\} \Rightarrow (10\text{-}38)$

Corollary. The power spectrum of an arbitrary process $\mathbf{x}(t)$ real or complex is nonnegative:

$$\boxed{S(\omega) \geq 0} \tag{10-39}$$

Proof (*Indirect*). Suppose that $S(\omega)$ is negative for $\omega = \omega_0$:

$$S(\omega_0) < 0$$

We can then find a small enough interval (ω_1,ω_2) near ω_0 such that

$$S(\omega) < 0 \qquad \text{for } \omega_1 < \omega < \omega_2$$

† For numerical details, see W. W. Solodownikow and A. S. Uskow, "Statistische Analyse von Regelstrecken," VEB Verlag Technik, Berlin, 1963.

‡ See "The Fourier Integral," p. 16.

3.5.1 Stochastic Differential Equations Driven by White Noise

Continuous-time

$$\dot{x} = F(t)x + G(t)w + C(t)u$$
$$z = H(t)x + v + D(t)u$$

$x = x(t)$ = state vector
$z = z(t)$ = measurement vector
$u = u(t)$ = deterministic input vector (control)
$w = w(t)$ = system (plant) zero-mean white noise (Gaussian)
$v = v(t)$ = measurement zero-mean white noise (Gaussian)

$$E(w(t)) = 0, \quad E(v(t)) = 0$$
$$E(w(t_1)\,w^T(t_2)) = Q(t_1)\,\delta(t_2 - t_1)$$
$$E(v(t_1)\,v^T(t_2)) = R(t_1)\,\delta(t_2 - t_1)$$
$$E(w(t_1)\,v^T(t_2)) = M(t_1)\,\delta(t_2 - t_1)$$

correlation between system and measurement white noise (usually $= 0$ no correlation)

Also

$$\left.\begin{array}{l} E(x(t_0)) = \bar{x}_0 \\ E\big((x(t_0) - \bar{x}_0)(x(t_0) - \bar{x}_0)^T\big) \end{array}\right\}$$

mean and variance of initial state must be known

These are the initial conditions for the above differential equation

$\xrightarrow{\text{Gaussian}}$ need to find only the first and second moments for $x(t)$

3.5.2 Discrete-Time SDEquations Driven by White Noise

$$x_k = \Phi_k x_{k-1} + G_{k-1} w_{k-1} + \Gamma_{k-1} u_{k-1}$$
$$z_k = H_k x_k + v_k + D_k u_k$$

$$E(w_k) = 0, \qquad E(v_k) = 0$$
$$E(w_{k_1} w_{k_2}^T) = Q_{k_1} \Delta(k_2 - k_1)$$
$$E(v_{k_1} v_{k_2}^T) = R_{k_1} \Delta(k_2 - k_1)$$
$$E(w_{k_1} \cdot v_{k_2}^T) = M_{k_1} \Delta(k_2 - k_1)$$

$$\Delta(k_2 - k_1) = \begin{cases} 1 & k_1 = k_2 \\ 0 & k_1 \neq k_2 \end{cases}$$

Initial conditions for x_{k_0}, represented by mean and variance are given

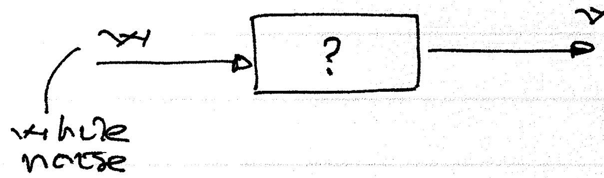
$\xRightarrow{\text{Gaussian}}$ need to find only the first and second moments for x_k

3.6 SHAPING FILTERS

$$(1) \quad \begin{aligned} \dot{x} &= Fx + G w_1 \\ z &= Hx + v \end{aligned}$$

w_1 not a white noise process

v = white noise



Problem: find the system driven by white noise whose output is w_1

Let

$$(2) \quad \begin{aligned} \dot{x}_{SF} &= F_{SF} x_{SF} + G_{SF} w \\ w_1 &= H_{SF} x_{SF} \end{aligned}$$

Let us augment (1) and (2)

$$\underline{X}(t) = \begin{bmatrix} x(t) \\ x_{SF}(t) \end{bmatrix}$$

$\Rightarrow$

$$\begin{bmatrix} \dot{x} \\ \dot{x}_{SF} \end{bmatrix} = \begin{bmatrix} F & G H_{SF} \\ 0 & F_{SF} \end{bmatrix} \begin{bmatrix} x \\ x_{SF} \end{bmatrix} + \begin{bmatrix} 0 \\ G_{SF} \end{bmatrix} w$$

$$z = [H \quad 0] \begin{bmatrix} x \\ x_{SF} \end{bmatrix} + v$$

or

$$\dot{\underline{X}} = F_T \underline{X} + G_T w$$

$$z = H_T \underline{X} + v$$

The problem is how to find F_{SF}, G_{SF}, H_{SF} (matrices for shaping filter). Solution is not unique. One example is given in Example 3.5, page 75.

MEASUREMENT NON-WHITE NOISE

$$\dot{x} = Fx + Gw \qquad w = \text{white noise}$$

$$z = Hx + v_1 \qquad v_1 = \text{non-white noise}$$

$$\dot{x}_{SF} = F_{SF} x_{SF} + G_{SF} v \qquad v = \text{white noise}$$

$$v_1 = H_{SF} x_{SF}$$

$$\begin{bmatrix} \dot{x} \\ \dot{x}_{SF} \end{bmatrix} = \begin{bmatrix} F & 0 \\ 0 & F_{SF} \end{bmatrix} \begin{bmatrix} x \\ x_{SF} \end{bmatrix} + \begin{bmatrix} G & 0 \\ 0 & G_{SF} \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix}$$

$$z = [H \quad H_{SF}] \begin{bmatrix} x \\ x_{SF} \end{bmatrix}$$

note: no noise in measurements