Chapter 3 Stochastic Processes

Jan. 31, 97 (2)

3.1

3.2 Probability and Random Variables

The set of all possible outcomes of a statistical experiment is called a SAMPLE SPACE, denoted by S.

Example: The tossing of a die (a cube with six faces produces six outcomes: $O_a, O_b, O_c, O_d, O_e, O_f$.

Relative frequencies of outcomes of a statistical experiment are called PROBABILITIES

Events are collections of outcomes on which probabilities are assigned

$$p(\mathcal{E}) = \frac{\#(\mathcal{E})}{\#(S)}$$

$\#(\mathcal{E})$ = number of outcomes in an event
$\#(S)$ = number of outcomes in a sample space

ASIDE: Axioms

$$\begin{cases} p(\mathcal{E}) \geq 0 \\ p(S) = 1 \\ \text{If } \mathcal{E}_1 \cdot \mathcal{E}_2 = 0 \Rightarrow p(\mathcal{E}_1 + \mathcal{E}_2) = p(\mathcal{E}_1 + \mathcal{E}_2) \end{cases}$$

A <u>random variable</u> assigns real numbers to outcomes, $d: S \to \mathbb{R}$. RV is a real function whose domain is S and range $\mathbb{R}$.

(Ex)

$d(O_a)=1, \quad d(O_b)=2, \quad d(O_c)=3$

$d(O_d)=4, \quad d(O_e)=5, \quad d(O_f)=6$

Probability Distribution

$$P_d(x) = p\{O \mid d(O) < x\}$$

note that for every real x the set $\{O \mid d(O) < x\}$ is an event.

ASIDE: Properties of $P_d(x)$

$P_d(-\infty) = 0$

$P_d(+\infty) = 1$

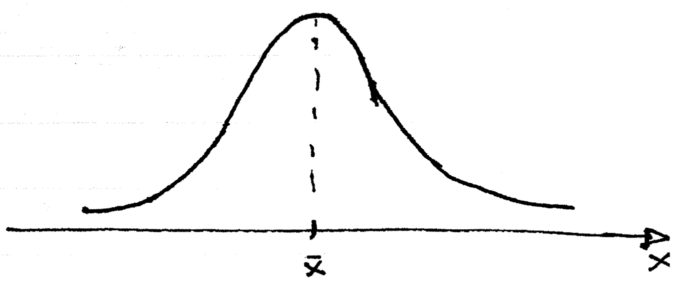
$P_d(x_1) \leq P_d(x_2)$ for $x_1 < x_2$

The Density Function

$$p_d(x) = \frac{d}{dx} P_d(x)$$

Gaussian density function:

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-\bar{x})}{\sigma}}$$



$\bar{x} = E\langle x\rangle$ = mean

σ^2 = variance

$$\Rightarrow \underbrace{N}_{\text{normal}}(\underbrace{\bar{x}}_{\text{mean}}, \overbrace{\sigma^2}^{\text{variance}})$$

Generating Function (characteristic function) for Gauss.

$$\mathcal{F}^{-1}(p(x)) = \underbrace{\frac{1}{\sqrt{2\pi}}}_{\text{scaled}} \int_{-\infty}^{+\infty} p(x)\, e^{i\omega x}\, dx = p(\omega) = \frac{\sigma}{\sqrt{2\pi}} e^{-\frac{1}{2}\omega^2\sigma^2}$$

Vector-valued Gaussian Density Function

$$p(x) = \frac{1}{\sqrt{\det P\,(2\pi)^n}}\, e^{-\frac{1}{2}(x-\bar{x})^T P^{-1}(x-\bar{x})}$$

and

$$p(\omega) = \frac{1}{\sqrt{(2\pi)^{-n}\det P^{-1}}} \cdot e^{-\frac{1}{2}\omega^2 P \omega}$$

where

P is the covariance matrix (to be defined)

Joint Probability

$$p(\varepsilon_1 \cap \varepsilon_2) =$$

$$= p(\varepsilon_1 \cdot \varepsilon_2) \overset{\text{If}}{=} p(\varepsilon_1)\cdot p(\varepsilon_2) \Rightarrow \text{independent events}$$

Conditional Probability

$$p(\varepsilon_1 | \varepsilon_2) = \frac{p(\varepsilon_1 \cap \varepsilon_2)}{p(\varepsilon_2)}$$

3.3 STATISTICAL PROPERTIES OF RANDOM VARIABLES

Moments

$$\eta_n(x) = E\langle x^n \rangle = \int_{-\infty}^{+\infty} x^n \, p(x) \, dx$$

$n=1 \Rightarrow \quad \eta_1(x) = \text{mean} = \text{expected value}$

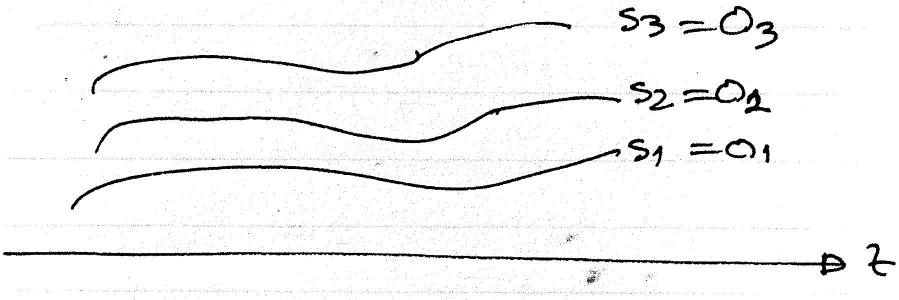
Central Moments

$$\mu_n(x) = E\langle x - \eta_1(x) \rangle^n = E\langle x - E\langle x \rangle \rangle^n = \int_{-\infty}^{+\infty} \left(x - E(x)\right)^n p(x) \, dx$$

$n=2 \Rightarrow$ 2nd central moment = variance

3.4 STATISTICAL PROPERTIES OF RANDOM PROCESSES

$x(t, s)$ — t: time, s: outcome



stochastic process is a family of time functions parametrized by outcomes.

Vector stochastic process

$$x(t,s) = \begin{bmatrix} x_1(t,s) \\ x_2(t,s) \\ \vdots \\ x_n(t,s) \end{bmatrix} \qquad x_i(t,s) = \text{scalar stochastic processes}$$

Mean of a Vector Stochastic Process

$$E(x(t)) = \begin{bmatrix} E(x_1(t)) \\ E(x_2(t)) \\ \vdots \\ E(x_n(t)) \end{bmatrix}, \quad E(x_i(t)) = \int_{-\infty}^{+\infty} x_i(t)\, p(x_i(t))\, dx_i(t)$$

note that everything is function of time.

Correlation Matrix

$$E\left(x(t_1)\, x^T(t_2)\right) = E \begin{bmatrix} x_1(t_1)\, x_1(t_2) & \cdots & x_1(t_1)\, x_n(t_2) \\ \vdots & & \\ x_n(t_1)\, x_1(t_2) & \cdots & x_n(t_1)\, x_n(t_2) \end{bmatrix}$$

Covariance Matrix

$$E\left\{ (x(t_1) - \underbrace{E(x(t_1))}_{\eta(t_1)})(x(t_2) - \underbrace{E(x(t_2))}_{\eta(t_2)})^T \right\}$$

$$= E \begin{bmatrix} (x_1(t_1) - \eta_1(t_1))(x_1(t_2) - \eta_1(t_2)) & \cdots & (x_1(t_1) - \eta_1(t_1))(x_n(t_2) - \eta_n(t_2)) \\ \vdots & & \\ (x_n(t_1) - \eta_n(t_1))(x_1(t_2) - \eta_1(t_2)) & \cdots & (x_n(t_1) - \eta_n(t_1))(x_n(t_2) - \eta_n(t_2)) \end{bmatrix}$$

for $t_1 = t_2 \Rightarrow$ Covariance Matrix = Variance Matrix

Cross-Covariance

$$E\left\{ (x(t_1) - \eta_x(t_1))\,(y(t_2) - \eta_y(t_2)) \right\}$$

3.4.3 White Noise

Uncorrelated stochastic processes

$$E\{(x(t_1)-E(x(t_1)))(y(t_2)-E(y(t_2)))^T\} = 0$$

$\Rightarrow$ $x(t)$ and $y(t)$ are uncorrelated for all t_1 and t_2

Cross-covariance $= 0 \Rightarrow$ uncorrelated

Orthogonal stochastic processes

$$E\{x(t_1)\cdot y^T(t_2)\} = 0$$

correlation matrix $= 0 \Rightarrow$ orthogonal

Uncorrelated stochastic process

$$E\{(x(t_1)-E(x(t_1)))(x(t_2)-E(x(t_2)))^T\} = \underbrace{Q(t_1,t_2)\,\delta(t_1-t_2)}_{\text{delta function}}$$

Independent stochastic process

if $x(t_1), x(t_2), \ldots, x(t_n)$ are independent

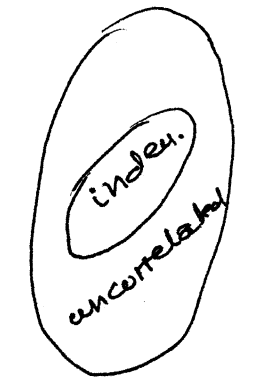
$$\Rightarrow p_{x(t_1),\ldots,x(t_n)}(s_1, s_2, \ldots, s_n) = p_{x(t_1)}(s_1)\cdot p_{x(t_2)}(s_2)\cdots p_{x(t_n)}(s_n)$$

In general

independent $\Longrightarrow$ uncorrelated

independent $\not\Longleftarrow$ uncorrelated

no the other way around



For Gaussian

independent $\Longleftrightarrow$ uncorrelated

Stationary (time invariant) stochastic processes

$$p(x_1, x_2, \ldots, x_n, t_1, t_2, \ldots, t_n) = p(x_1, x_2, \ldots, x_n, t_1+\varepsilon, t_2+\varepsilon, \ldots, t_n+\varepsilon)$$

$\Rightarrow$ stationary

Wide-sense stationary stochastic process

1) $E(x(t)) = \text{const}$

2) $E(x(t_1)\, x^T(t_2)) = Q(t_2 - t_1) = Q(\tau)$

Ergodic Random Processes

A process is ergodic if its statistics can be determined from a single function $x(t, \xi)$ (from a single observation)

Markov Processes and Sequences

If

$$p\{x(t_i) \mid x(\tau);\ \tau < t_{i-1}\} = p\{x(t_i) \mid x(t_{i-1})\}$$

$\Rightarrow$ Markov process (continuous-time)

$$p\{x_i \mid x_k;\ k \le i-1\} = p\{x_i \mid x_{i-1}\}$$

$\Rightarrow$ Markov sequence (discrete-time)

Note that solutions of the first order vector linear differential and difference equations are Markov processes

$$x_k = \Phi_{k-1} x_{k-1} + G_{k-1} w_{k-1}$$

For Gaussian stochastic processes

wide-sense stationary = stationary

orthogonal = independent

all statistics completely determined by the first- and second-order moments

Power Spectral Density (PSD)

Autocorrelation

$$\psi_x(\tau) = E\left(x(t)\,x(t+\tau)\right)$$

Its Fourier transform is the PSD, that is

$$\Psi_x(\omega) = \mathcal{F}\left(\psi_x(\tau)\right) = \int_{-\infty}^{+\infty} \psi_x(\tau)\, e^{-j\omega\tau}\, d\tau$$

and

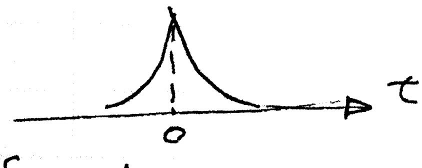
$$\psi_x(\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Psi_x(\omega)\, e^{j\omega\tau}\, d\omega$$

Properties:

1) Autocorrelation function is an even function

2) Its maximum is at origin $\max_{\tau} \psi_x(\tau) = \psi_x(0)$

$$\psi_x(0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Psi_x(\omega)\, d\omega$$



3) The PSD is nonnegative even function

$$\Psi_x(\omega) = \Psi_x(-\omega) \geq 0, \quad \forall \omega$$

Cross Power Spectral Density

$$\psi_{xy}(\tau) = E\left(x(t)\,y(t+\tau)\right) \Rightarrow \Psi_{xy}(\omega) = \int_{-\infty}^{+\infty} \psi_{xy}(\tau)\, e^{-j\omega\tau}\, d\tau$$