$\boxed{6.4.4}$ Symmetric Square Roots of Elementary Matrices

ELEMENTARY MATRICES

$$\underbrace{I}_{\text{identity}} - \underbrace{s}_{\text{scalar}} \overbrace{u v^T}^{\text{vectors}}$$

<u>products of elementary matrices are elementary matrices</u>

symmetric elementary matrices for $u = v$

$I - \sigma v v^T$ is symmetric

$$(I - \sigma v v^T)^2 = (I - \sigma v v^T)(I - \sigma v v^T)$$

$$= I - \sigma v v^T - \sigma v v^T + \sigma^2 v \underbrace{v^T v}_{\text{scalar} = |v|^2} v^T$$

$$= I - 2\sigma v v^T + \sigma^2 |v|^2 v v^T$$

$$= I - \underbrace{(2\sigma - \sigma^2 |v|^2)}_{= s = \text{scalar}} v v^T$$

or the other way around

$$(I - s v v^T)^{1/2} = I - \sigma v v^T$$

$$s = 2\sigma - \sigma^2 |v|^2 \Rightarrow \sigma = \frac{1 + \sqrt{1 - s|v|^2}}{|v|^2}$$

which is a method for finding the square root of symmetric elementary matrices

Note that we need

$1 - s|v|^2 \geq 0$ for the existence of the square root

(6.4.5) TRIANGULATION METHODS

known as QR-decomposition in linear numerical algebra

has nothing to do with our Q and R matrices

$$M = Q \cdot R \sim \text{orthogonal matrix } (R^T = R^{-1})$$

Q: triangular matrix

Let

$$M = CC^T \qquad \text{with} \qquad C = Q \cdot R$$

then

$$M = QR \cdot (QR)^T = Q \underbrace{R \cdot R^T}_{=I} \cdot Q^T = QQ^T \Rightarrow$$

NEED ONLY TO FIND TRIANGULAR FACTOR

this can be used in Kalman filtering for updates of Cholesky factors

TRIANGULARIZATION BY GIVENS ROTATIONS

$$T_{ij} = \begin{array}{c} \\ \left[\begin{array}{cccccccc} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & \cos\theta & & \sin\theta & & \\ & & & & 1 & & & \\ & & & & & \ddots & & \\ & & & & & & 1 & \\ & & & -\sin\theta & & \cos\theta & & \\ & & & & & & 1 & \\ & & & & & & & \ddots \\ & & & & & & & & 1 \end{array}\right] \end{array} \begin{array}{c} \\ \\ \\ j \\ \\ \\ \\ i \\ \\ \\ \\ \end{array} \quad n \times n$$

$$\qquad\qquad\qquad j \qquad\qquad i$$

with all other elements equal to zero

T_{ij} is known as plane rotation matrix

How it works?

$$\overset{2\times3}{A} \cdot \overset{3\times3}{T_{23}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} a_{11} & * & * \\ a_{21} & 0 & * \end{bmatrix} = A_1$$

$$0 = a_{22}\cos\theta - a_{23}\sin\theta \Rightarrow \cos\theta = \frac{a_{23}}{\sqrt{a_{22}^2 + a_{23}^2}}, \quad \sin\theta = \frac{a_{22}}{\sqrt{a_{22}^2 + a_{23}^2}}$$

$$A_1 \cdot T_{23} = \begin{bmatrix} * & * & * \\ 0 & 0 & * \end{bmatrix} = A_2$$

$$A_2 \cdot T_{12} = \begin{bmatrix} 0 & * & * \\ 0 & 0 & * \end{bmatrix} = A_3$$

$$A_3 = A \cdot T_{23} \cdot T_{13} \cdot T_{12}$$

Givens rotations zero these elements one by one.

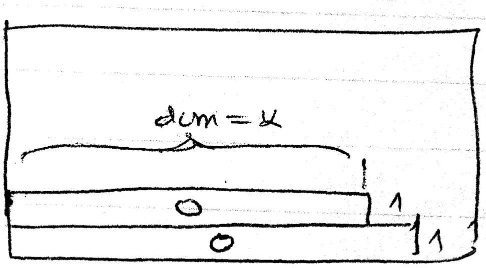
TRIANGULARIZATION by HOUSEHOLDER REFLECTIONS

$$T(v) = T = I - \frac{2}{v^T v} \cdot v v^T$$ this is an orthogonal elementary matrix

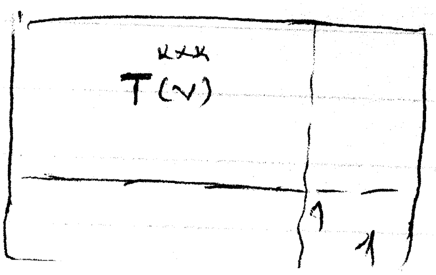
$$T T^T = \left(I - \frac{2}{v^T v} v v^T\right)\left(I - \frac{2}{v^T v} v v^T\right)$$

$$= I - \frac{4}{v^T v} v v^T + \frac{4}{(v^T v)^2} v (v^T v) v^T = I$$

A single householder reflection can be used to zero all the elements to the left of the diagonal in an entire row of a matrix.

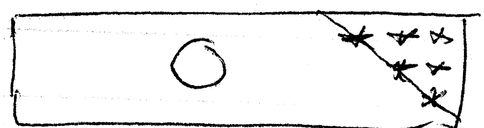


The size of the Householder transformation shrinks with each step



The result

$A \, T(v_1) \, T(v_2) \cdots T(v_m)$



Householder triangulation algorithm is given on page 234.
Example 6.10 shows how to choose vectors v_i

6.5 FACTORED IMPLEMENTATIONS OF THE OBSERVATIONAL UPDATE

6.5.1 Potter SQUARE Root FACTORIZATION

let
$$P(-) \stackrel{def}{=} C(-)C^T(-)$$
$$P(+) \stackrel{def}{=} C(+)C^T(+)$$
then the observational update equation
$$P(+) = P(-) - P(-)H^T(HP(-)H^T+R)^{-1}HP(-)$$
could be factored as
$$C(+)C^T(+) = C(-)C^T(-) - C(-)C^T(-)H^T[HC(-)C^T(-)H^T+R]^{-1}HC(-)C^T(-)$$

Introduce $V = C^T(-)H$

$$= C(-)C^T(-) - C(-)V[V^TV+R]^{-1}V^TC^T(-)$$
$$= C(-)[I - V(V^TV+R)^{-1}V^T)C^T(-)$$

For the case of scalar measurements Potter got factorization of
$$(*) \quad I - V(V^TV+R)^{-1}V^T = WW^T$$

Hence
$$C(+)C^T(+) = C(-)\{WW^T\}C^T(-)$$
$$= \{C(-)W\}\{C(-)W\}^T$$

$\Rightarrow$

$$\boxed{C(+) = C(-)W}$$

For scalar measurement the expression to be factored (*) is a symmetric elementary matrix
$$I - \frac{vv^T}{R+|v|^2}, \quad v = C^T(-)H^T \text{ is a column } n\text{-vector}$$

From the formula for symmetric square root of elementary matrices (6.45 in the book)

$$s = \frac{1}{R+|v|^2}$$

$$\left(I - s v v^T\right)^{1/2} = I - \sigma v v^T$$

$$\sigma = \frac{1+\sqrt{1-s|v|^2}}{|v|^2} = \frac{1+\sqrt{\frac{R}{R+|v|^2}}}{|v|^2}$$

Note that

$$1 - s|v|^2 = 1 - \frac{|v|^2}{R+|v|^2} = \frac{R}{R+|v|^2} \geq 0$$

$\Rightarrow$ a real matrix square root always exists.

Since

$$C(+) = C(-)\,W$$

$$P(+) = C(+)\,C^T(+) = C(-)\,\underbrace{W\,W^T}_{= I - s v v^T}\,C^T(-)$$

$$= C(-)(I - \sigma v v^T)(I - \sigma v v^T)^T C^T(-)$$

$\Rightarrow$

$$C(+) = C(-)(I - \sigma v v^T)$$

with

$$\sigma = \frac{1+\sqrt{\frac{R}{R+|v|^2}}}{|v|^2}$$

} Potter square root observation update formula

Complexity (from Table 6.16) is

$$(3n^2 + 4n + 4)\text{ flops} + 1 \cdot \sqrt{\ } = O(3n^2)\text{ flops} + 1\sqrt{\ }$$

The algorithm from Table 6.16 updates the state estimates x and the Cholesky factor C.