Chapter 6 - Implementation Methods

6.1 "Computer roundoff errors can and do cause the Kalman filter to diverge."

"Robustness refers to the relative insensitivity of the solution to errors of some sort."
"Numerical stability refers to robustness against roundoff errors."

Well conditioned problem of the solution is not badly sensitive to the input data, and ill conditioned of the sensitivity is bad.

Ill-conditioned Kalman Filter

Factors that contribute to such ill-conditioning include the following:

1. Large uncertainties in the values of the matrix parameters Φ, Q, H, or R. Such modeling errors are not accounted for in the derivation of the Kalman filter.
2. Large ranges of the actual values of these matrix parameters, the measurements, or the state variables—all of which can result from poor choices of scaling or dimensional units.
3. Ill-conditioning of the intermediate result $R^* = HPH^T + R$ for inversion in the Kalman gain formula.
4. Ill-conditioned theoretical solutions of the matrix Riccati equation—without considering numerical solution errors. With numerical errors, the solution may become indefinite—which can destabilize the filter estimation error.
5. Large matrix dimensions. The number of arithmetic operations grows as the square or cube of matrix dimensions, and each operation can introduce roundoff errors.
6. Poor machine precision, which makes the roundoff errors larger.

There is a great difference between theory and practice.
Giacomo Antonelli (1806–1876)[1]

Computational complexity : how many flops are required for computations

Calculations involving the covariance matrix are a major cause of numerical difficulties in the conventional Kalman filter implementation.
≡ SOLUTION OF THE RICCATI EQUATION

Asymetry in calculation of the covariance matrix causes numerical unstability.

6.2

6.3 EARLIER IMPLEMENTATION METHODS

6.3.1 Swerling Formulation

$$P(+) = [\bar{P}^{-1}(-) + H^T R^{-1} H]^{-1} \qquad (1)$$

Let $\dim(x) = n$ and $\dim(z) = \ell$

computations in (1) require

$$O(2n^3 + \tfrac{1}{2}n^2\ell + n\ell^2 + \ell^3) \text{ flops}$$

for $n \gg \ell$ we have

$$O(2n^3) = O(n^3) \text{ flops for Swerling's Form}$$

NOTE from APPENDIX B, page 347 inversion formulas for matrix expressions

$$(A + BC^T)^{-1} = A^{-1} - A^{-1}B(A + C^TA^{-1}B)^{-1}C^TA^{-1}$$

and

$$(A + BC^{-1}D^T)^{-1} = A^{-1} - A^{-1}B(C + D^TA^{-1}B)^{-1}D^TA^{-1}$$

are used to relate different formulations of the Kalman filter

6.3.2 KALMAN FORMULATION

$$(2)\quad \begin{cases} K = P(-)H^T\left(HP(-)H^T+R\right)^{-1} = \{HP(-)\}^T\left[\{HP(-)\}H^T+R\right]^{-1} \\ P(+) = P(-) - K\{HP(-)\} \end{cases}$$

Note { } = common terms.

This formulation is of order of complexity

$$O\left(\tfrac{3}{2}n^2e + \tfrac{3}{2}ne^2 + e^3\right) \underset{\text{for } n \gg e}{=} O(n^2)$$

6.3.3 JOSEPH's Implementation

$$(3)\quad P(+) = (I-KH)P(-)(I-KH)^T + KRK^T \qquad \text{(see 4.23)}$$

Nice symmetry, but the complexity is

$$O(n^3)$$

6.4 FACTORIZATION METHODS FOR KALMAN FILTERING

6.4.2 CHOLESKY

(= error covariance + gain eqs)

Instead solving the Riccati equation for P ($P \geq 0$) solve it for C, where $CC^T = P$.

C is known as a Cholesky factor of P.

Basics about the Cholesky decomposition

Let $M \geq 0$ such that $M = CC^T$

C is either upper or lower triangular.

Modified Cholesky decomposition

$$M = UD_UU^T = LD_LL^T$$

U = upper triangular, L = lower triangular, D_L, D_U = diagonal

In general the Cholesky factor is not unique.

$$M = (CT)(CT)^T = CTT^TC^T$$

if T is orthogonal, that is, $TT^T = T^TT$ ($TT^T = TT^{-1} = I$)

$$\Rightarrow$$

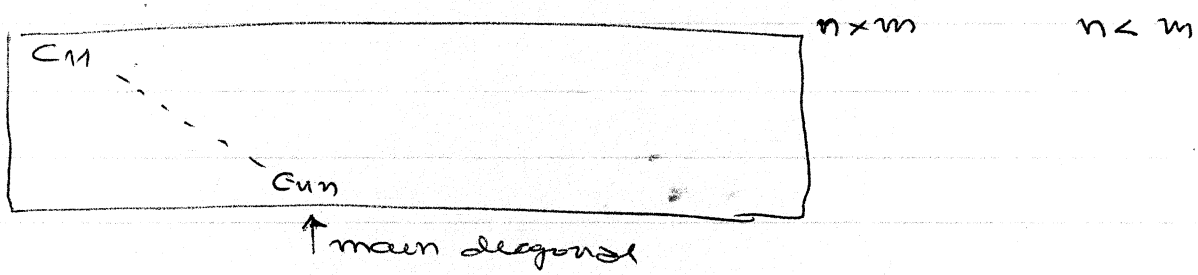
$$M = CC^T$$

Hence C and CT are Cholesky factors of M.

However, in applications to Kalman filtering we will assume that C is either upper or lower triangular with nonnegative diagonal elements, which implies the uniqueness C.

$$M^{n\times n} = C^{n\times m} \cdot (C^T)^{m\times n}$$

Main diagonal of $C^{n\times m}$ is a set of elements $\{c_{ii} \mid 1 \le i \le \min(n,m)\}$



Triangular matrices:

Upper triangular if nonzero elements are above the main diagonal

$$\begin{bmatrix} 1 & 3 & 4 & 5 & 6 \\ 0 & 2 & 7 & 8 & 9 \\ 0 & 0 & 3 & 2 & 1 \end{bmatrix}$$

Lower triangular: Non zero elements are below the main diagonal

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 0 \\ 3 & 2 \\ 4 & 5 \end{bmatrix}$$

How to get the desired Cholesky decomposition:

$$\underbrace{\begin{bmatrix} p_{11} & p_{21} \\ p_{21} & p_{22} \end{bmatrix}}_{\text{symmetric matrix } P} = \begin{bmatrix} c_{11} & 0 \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} c_{11} & 0 \\ c_{21} & c_{22} \end{bmatrix}^T$$

multiply and equate the corresponding elements which produces 3 eqs. for 3 unknowns

$$\begin{bmatrix} p_{11} & p_{21} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} c_{11} & 0 \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} c_{11} & c_{21} \\ 0 & c_{22} \end{bmatrix} = \begin{bmatrix} c_{11}^2 & c_{11}c_{21} \\ c_{21}c_{11} & c_{21}^2 + c_{22}^2 \end{bmatrix}$$

$$\Rightarrow \quad (1) \quad p_{11} = c_{11}^2 \quad \Rightarrow \quad c_{11} = \sqrt{p_{11}}$$

$$(2) \quad p_{21} = c_{11}c_{21} \quad \Rightarrow \quad c_{21} = p_{21}/c_{11}$$

$$(3) \quad p_{22} = c_{21}^2 + c_{22}^2 \qquad c_{22} = \sqrt{p_{22} - c_{21}^2} = \sqrt{p_{22} - p_{21}^2/p_{11}}$$

Complete Cholesky decomposition algorithm for a general $m \times m$ symmetric positive definite matrix is given in Table 6.7. Both upper and lower triangular decomposition algorithms are given. The same table gives the complexity result

$$\Rightarrow \frac{1}{6} m(m-1)(m+1) \text{ flops} + m \text{ square roots} \quad \Rightarrow \quad O\left(\frac{1}{6}m^3\right) \text{ flops} + m \text{ square roots}$$

Modified Cholesky Decomposition

$$M = U D U^T$$

D: diagonal matrix

U: unit upper triangular (all elements on the main diagonal = 1)

This is also known as UD decomposition.

The corresponding algorithm is given in Table 6.9. That table also gives the computational complexity as

$$\frac{1}{6}(n)(n-1)(n+4) \text{ flops} = O\left(\frac{1}{6}n^3\right) \text{ flops}$$

The main difference between the original Cholesky decomposition and the UD decomposition is that the UD decomposition does not require taking square roots

$$\begin{bmatrix} p_{11} & p_{21} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} 1 & u_{12} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} d_{11} & 0 \\ 0 & d_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ u_{12} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} d_{11} & u_{12} d_{22} \\ 0 & d_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ u_{12} & 1 \end{bmatrix} = \begin{bmatrix} d_{11} + d_{22} u_{12}^2 & u_{12} d_{22} \\ d_{22} u_{12} & d_{22} \end{bmatrix}$$

(1) $p_{22} = d_{22} \Rightarrow d_{22}$

(2) $p_{21} = u_{12} d_{22} \Rightarrow u_{12} = p_{21}/d_{22}$

(3) $p_{11} = d_{11} + d_{22} u_{12}^2 \Rightarrow d_{11} = p_{11} - d_{22} u_{12}^2$

We do have only products and divisions, but no need for $\sqrt{\ }$, which reduces the complexity by the factor of

$$n \times (\text{number of required computations for } \sqrt{\ })$$

Uses of Cholesky decomposition in Kalman filtering. Cholesky decomposition methods produce triangular matrix factors (Cholesky factors), and the sparseness of these factors can be exploited in the implementation of the Kalman filter equations. These methods are used for the following purposes:

1. In the decomposition of covariance matrices (P, R, and Q) for implementation of "square root" filters.
2. In "decorrelating" measurement errors between components of vector-valued measurements, so that the components may be processed sequentially as independent scalar-valued measurements. (See page 218.)
3. As part of a numerically stable method for computing matrix expressions containing the factor $(HPH^T+R)^{-1}$ in the conventional form of the Kalman filter. (This matrix inversion can be obviated by the decorrelation methods, however.)
4. In Monte Carlo analysis of Kalman filters by simulation. Cholesky factors are used for generating independent random sequences of vectors with prespecified means and covariance matrices. (See §3.4.8 on page 70.)

1. Bierman, Section 6.5

2. DECORRELATION OF VECTOR-VALUED MEASUREMENTS

Let $z = Hx + v$ $\qquad E\{v v^T\} = R\,\delta(t-\tau)$.

or $R\,\Delta(k-k_0)$ in discrete time

Since R is in general not diagonal matrix then the scalar components of z cannot be processed serially as scalar observations.

If we factor R as

$$R = UDU^T$$

and introduce the change of variables

$$z_{new} = U^{-1} z$$

$$= U^{-1}(Hx + v) = \underbrace{U^{-1}H}_{H_{new}} x + \underbrace{U^{-1}v}_{v_{new}}$$

$$z_{new} = H_{new}\, x + v_{new}$$

$$R_{new} = E\{v_{new}\, v_{new}^T\} \cdot \delta(\,-\,)$$

$$= U^{-1} E\{v v^T\} (U^T)^{-1} \delta(\,-\,)$$

$$= U^{-1} R U^{-T} = U^{-1}(UDU^T) U^{-T} \delta(\,-\,) = D\,\delta(\,-\,)$$

Thus,

$$\boxed{R_{new} = D} \qquad \text{from} \quad R = UDU^T$$

or $H_{new} = U^{-1}H$

$$\boxed{U H_{new} = H} \Rightarrow$$ need to solve the unit upper triangular system of algebraic equations

It requires only the back substitution part of the Gaussian elimination

Table 6.10 $\Rightarrow$

$$U^{m \times m} H_{new}^{m \times p} = H^{m \times p}$$

$\Rightarrow p \cdot \frac{m(m-1)}{2}$ flops or $O\left(\frac{m^2}{2}\right)$ flops per column

Table 6.11 $\Rightarrow$

The measurement decorrelation procedure requires

$$\frac{\ell(\ell-1)(\ell+4)}{6} + \frac{\ell(\ell-1)(n+1)}{2} \text{ flops}$$

where $R^{\ell \times \ell}$.

3. COMPUTING $(HPH^T + R)^{-1}$ by using CHOLESKY

Let $(HPH^T + R) = UDU^T$

Then, in order to calculate

$$(HPH^T + R)^{-1} \cdot H$$

Solve

$$\underbrace{(UDU^T)\underbrace{(HPH^T + R)^{-1}}_{=I} \cdot H}_{=X} = H$$

$$UDU^T X = H$$

$$\underbrace{U D U^T}_{} x = H$$

$$= x_{[1]}$$

$$\overset{e \times e}{U} \overset{e \times n}{x_{[1]}} = \overset{e \times n}{H} \Rightarrow \overset{e \times n}{x_{[1]}} \quad \text{in} \quad n \frac{e(e-1)}{2} \text{ flops}$$

$$\underbrace{D U^T x}_{x_{[2]}} = x_{[1]}$$

$$\overset{e \times e}{D} \overset{}{x_{[2]}} = x_{[1]} \Rightarrow \overset{e \times n}{x_{[2]}} \quad \text{in} \quad n \cdot e \text{ flops}$$

$$U^T x = x_{[2]} \Rightarrow \overset{e \times n}{x} \quad \text{in} \quad n \cdot \frac{e(e-1)}{2} \text{ flops}$$

$$\Downarrow$$

$$\approx n \frac{e^2}{2} + n \frac{e^2}{2} = n e^2 \text{ flops}$$

6.4.3

Implementation of the Kalman filter with "decorrelated" measurements is discussed in Kaminski, 1971, Ph.D thesis Stanford.

TABLE 6.15 OPERATIONS FOR SEQUENTIAL PROCESSING OF MEASUREMENTS

Operation	flops
$H \times P(-)$	n^2
$H \times (HP(-))^T + R$	n
$\left[H (HP(-))^T + R\right]^{-1}$	1
$\left[H (HP(-))^T + R\right]^{-1} \times (HP(-))$	n
$P(-) - (HP(-)) \times \left[H (HP(-))^T + R\right]^{-1} (HP(-))$	$\frac{1}{2}n^2 + \frac{1}{2}n$
Total (per component) $\times \ell$ components	$\left\{\frac{3}{2}n^2 + \frac{5}{2}n + 1\right\} \times \ell$
+ decorrelation complexity	$\frac{2}{3}\ell^3 + \ell^2 - \frac{5}{3}\ell + \frac{1}{2}\ell^2 n - \frac{1}{2}\ell n$
Total	$\frac{2}{3}\ell^3 + \ell^2 - \frac{2}{3}\ell + \frac{1}{2}\ell^2 n + 2\ell n + \frac{3}{2}\ell n^2$

$$= O\left(\tfrac{3}{2} e n^2 + \tfrac{2}{3} e^3 + \tfrac{1}{2} e^2 n\right)$$

compared to the complexity of conventional Kalman filter

$$= O\left(\tfrac{3}{2} e n^2 + e^3 + \tfrac{3}{2} e^2 n\right)$$