(5.6) DISCRETE LINEARIZED KALMAN FILTER

TABLE 5.3 DISCRETE LINEARIZED KALMAN FILTER EQUATIONS

Nonlinear nominal trajectory model:

$$x_k^{\text{NOM}} = f(x_{k-1}^{\text{NOM}}, k-1)$$

Linearized perturbed trajectory model:

$$\begin{aligned}
\delta x &\stackrel{\text{def}}{=} x - x^{\text{NOM}} \\
\delta x_k &\approx \left.\frac{\partial f(x,k-1)}{\partial x}\right|_{x=x_{k-1}^{\text{NOM}}} \delta x_{k-1} + w_k \\
w_k &\sim N(0, Q_k)
\end{aligned}$$

Nonlinear measurement model:

$$\begin{aligned}
z_k &= h(x_k, k) + v_k \\
v_k &\sim N(0, R_k)
\end{aligned}$$

Linearized approximation equations

Linear perturbation prediction:

$$\begin{aligned}
\widehat{\delta x}_k(-) &= \Phi_{k-1}^{[1]} \widehat{\delta x}_{k-1}(+) \\
\Phi_{k-1}^{[1]} &\approx \left.\frac{\partial f(x,k-1)}{\partial x}\right|_{x=x_{k-1}^{\text{NOM}}}
\end{aligned}$$

Conditioning the predicted perturbation on the measurement:

$$\begin{aligned}
\widehat{\delta x}_k(+) &= \widehat{\delta x}_k(-) + \bar{K}_k \left[\underbrace{z_k - h_k(x_k^{\text{NOM}})}_{=\delta z_k} - H_k^{[1]} \widehat{\delta x}_k(-)\right] \\
H_k^{[1]} &\approx \left.\frac{\partial h(x,k)}{\partial x}\right|_{x=x_k^{\text{NOM}}}
\end{aligned}$$

Computing the *a priori* covariance matrix:

$$P_k(-) = \Phi_{k-1}^{[1]} P_{k-1}(+) \Phi_{k-1}^{[1]T} + Q_{k-1}$$

Computing the Kalman gain:

$$\bar{K}_k = P_k(-) H_k^{[1]T} \left[H_k^{[1]} P_k(-) H_k^{[1]T} + R_k\right]^{-1}$$

Computing the *a posteriori* covariance matrix:

$$P_k(+) = \left\{I - \bar{K}_k H_k^{[1]}\right\} P_k(-)$$

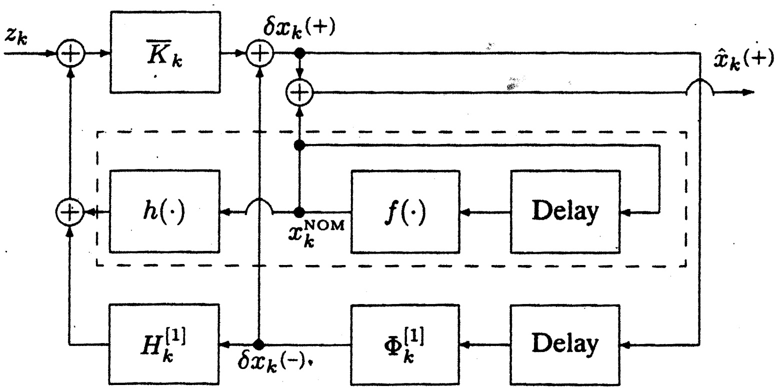

Figure 5.1 Estimator linearized about a "nominal" state.

(1)

5.7 DISCRETE EXTENDED KALMAN FILTER

TABLE 5.4 DISCRETE EXTENDED KALMAN FILTER EQUATIONS

Nonlinear dynamic model:

$$x_k = f(x_{k-1}, k-1) + w_{k-1}$$
$$w_k \sim N(0, Q_k)$$

Nonlinear measurement model:

$$z_k = h(x_k, k) + v_k$$
$$v_k \sim N(0, R_k)$$

Nonlinear implementation equations

Computing the predicted state estimate:

$$\hat{x}_k(-) = f(\hat{x}_{k-1}^{(+)}, k-1)$$

Computing the predicted measurement:

$$\hat{z}_k = h(\hat{x}_k(-), k)$$

Linear approximation equations

$$\Phi_{k-1}^{[1]} \approx \left.\frac{\partial f(x, k-1)}{\partial x}\right|_{x=\hat{x}_{k-1}^{(-)}}$$

Conditioning the predicted estimate on the measurement:

$$\hat{x}_k(+) = \hat{x}_k(-) + \bar{K}_k\,(z_k - \hat{z}_k)$$

$$H_k^{[1]} \approx \left.\frac{\partial h(x, k)}{\partial x}\right|_{x=\hat{x}_k(-)}$$

Computing the *a priori* covariance matrix:

$$P_k(-) = \Phi_{k-1}^{[1]} P_{k-1}(+) \Phi_{k-1}^{[1]T} + Q_{k-1}$$

Computing the Kalman gain:

$$\bar{K}_k = P_k(-) H_k^{[1]T} \left[H_k^{[1]} P_k(-) H_k^{[1]T} + R_k \right]^{-1}$$

Computing the *a posteriori* covariance matrix:

$$P_k(+) = \left\{ I - \bar{K}_k H_k^{[1]} \right\} P_k(-)$$

5.8 CONTINUOUS LINEARIZED and EXTENDED KALMAN FILTER

TABLE 5.5 CONTINUOUS EXTENDED KALMAN FILTER EQUATIONS

Nonlinear dynamic model:

$$\dot{x}(t) = f\big(x(t), t\big) + G(t)w(t)$$
$$w(t) \sim \mathcal{N}(0, Q(t))$$

Nonlinear measurement model:

$$z(t) = h\big(x(t), t\big) + v(t)$$
$$v(t) \sim \mathcal{N}\big(0, R(t)\big)$$

Implementation equations

Differential equation of the state estimate:

$$\dot{\hat{x}}(t) = f\big(\hat{x}(t), t\big) + \bar{K}(t)\,[z(t) - \hat{z}(t)]$$

Predicted measurement:

$$\hat{z}(t) = h\big(\hat{x}(t), t\big)$$

Linear approximation equations:

$$F^{[1]}(t) \approx \left.\frac{\partial f(x(t), t)}{\partial x}\right|_{x=\hat{x}(t)}$$

$$H^{[1]}(t) \approx \left.\frac{\partial h(x(t), t)}{\partial x}\right|_{x=\hat{x}(t)}$$

Kalman gain equations:

$$\dot{P}(t) = F^{[1]}(t)P(t) + P(t)F^{[1]T}(t) + G(t)Q(t)G^T(t) - \bar{K}(t)R(t)\bar{K}^T(t)$$
$$\bar{K}(t) = P(t)H^{[1]T}(x, t)R^{-1}(t)$$

From Friedland
"Advanced Control
System Design"
Prentice Hall, 1993

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$$A = [\partial f/\partial x]_{x=\hat{x}}$$
$$C = [\partial g/\partial x]_{x=\hat{x}}$$

for the nonlinear process defined by

$$\dot{x} = f(x, u) + Gv$$
$$y = g(x, u) + w$$

$$\dot{P} = AP + PA' - PC'W^{-1}CW + GVG'$$

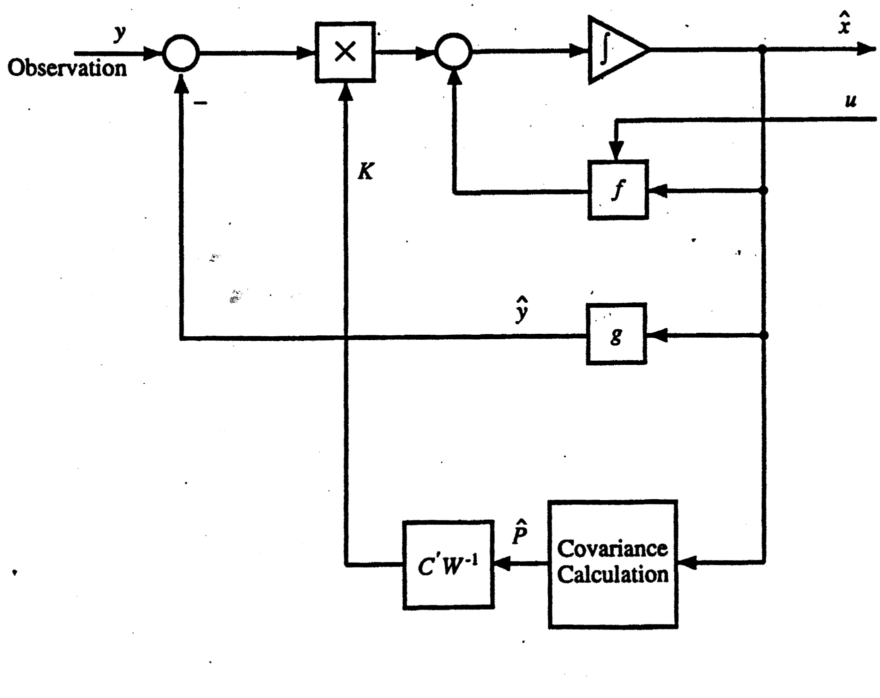

Figure 6.19 Schematic of extended Kalman filter, showing coupling between state estimation and covariance computation.

(5.9)

(5.10) APPLICATION OF NONLINEAR FILTERING

(Example 5.3) Damping parameter estimation

From Example 4.3 we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\zeta\omega \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w + \begin{bmatrix} 0 \\ 12 \end{bmatrix}$$

$$z = x_1 + v$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad P(0) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$Q = 4.47$, $R = 0.01$,

<u>Problem:</u> ζ = damping coefficient is unknown but constant

Let $x_3 = \zeta \quad \Rightarrow \quad \dot{x}_3 = 0$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\omega^2 x_1 - 2 x_2 x_3 \omega \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} w + \begin{bmatrix} 0 \\ 12 \\ 0 \end{bmatrix}$$

$$P(0) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad Q, R \text{ unchanged}$$

Chapter 5

Nonlinear Applications

5.1 A scalar stochastic sequence x_k is given by

$$
\begin{aligned}
x_k &= -.1\, x_{k-1} + \cos\, x_{k-1} + w_{k-1} \\
z_k &= x_k^2 + v_k \\
Ew_k = 0 &= Ev_k \\
\mathrm{cov} w_k &= \Delta(k_2 - k_1) \\
\mathrm{cov} v(t) &= 0.5\Delta(k_2 - k_1) \\
x_0 &\ , \ \ Ex_0 = 0 \\
P_0 &= 1 \\
x_k^{\mathrm{NOM}} &= 1
\end{aligned}
$$

Determine the linearized and extended Kalman estimator equations.

SOLUTION TO EXERCISE 5.1 (a) Linearized Kalman Filter:

$$
\begin{aligned}
\Phi^{[1]}(X_k^{\mathrm{NOM}}) &= \frac{\partial}{\partial x} f_{k-1}|_{x=x_{k-1}^{\mathrm{NOM}}} \\
&= [-0.1 - \sin x_{k-1}]|_{x=x_{k-1}^{\mathrm{NOM}}=-0.1-\sin 1=-0.9415} \\
H^{[1]}(X_k^{\mathrm{NOM}}) &= \frac{\partial}{\partial x}[x_k^2]|_{x=x_k^{\mathrm{NOM}}=2x_k^{\mathrm{NOM}}=2} \\
\hat{x}_k(+) &= \hat{\delta} x_k(+) + 1 && (5.92) \\
\hat{\delta} x_k(+) &= -0.9415\ \ \hat{\delta} x_{k-1}(+) + \bar{k}_k[Z_k - 1 + 1.883\ \ \hat{\delta} x_{k-1}(+)] && (5.93) \\
P_k(-) &= 0.8864\ \ P_k(+) + 1 && (5.94) \\
P_k(+) &= [1 - 2\bar{k}_k]\ \ P_k(-) r && (5.95) \\
\bar{k}_k &= 2P_k(-)\ /\ [4\ P_k(-) + 0.5] && (5.96)
\end{aligned}
$$

(b) Extended Kalman Filter:

$$
\Phi_{k-1}^{[1]} = \frac{\partial}{\partial x} f_k \Big\|_{x=\hat{x}_{k-1}}
$$

$$
\begin{aligned}
&= -0.1 - \sin \hat{x}_{k-1} \\
H_k^{[1]} &= \left.\frac{\partial}{\partial x} h_k \right|_{x=\hat{x}_k} \\
&= 2\hat{x}_k \\
\hat{x}_k(+) &= -0.1\hat{x}_{k-1}(+) + \cos \hat{x}_{k-1}(+) + \bar{k}_k[Z_k - (\hat{x}_k)^2] && (5.97) \\
P_k(-) &= (0.1 + \sin \hat{x}_{k-1}(-))^2 \;\; P_{k-1}(+) + 1 && (5.98) \\
\bar{k}_k &= 2\hat{x}_k(-) \;\; P_k(-) \;\; / \;\; [4(\hat{x}_k(-1))^2 \;\; P_k(-) + 0.5] && (5.99) \\
P_k(+) &= [1 - 2\hat{x}_k(-)\bar{k}_k] \;\; P_k(-) && (5.100) \\
& && (5.101)
\end{aligned}
$$

5.2 A scalar stochastic process $x(t)$ is given by

$$
\begin{aligned}
\dot{x}(t) &= -0.5x^2(t) + w(t) \\
z(t) &= x^3(t) + v(t) \\
\mathrm{E}\langle w(t) \rangle &= \mathrm{E}\langle v(t) \rangle = 0 \\
\mathrm{cov}.w_t &= \delta(t_1 - t_2) \quad , \; \mathrm{cov}.v(t) = 0.5\delta(t_1 - t_2) \\
x_0 \quad &, \quad Ex_0 = 0 \\
P_0 &= 1 \;\; .x_k^{\mathrm{NOM}} = 1
\end{aligned}
$$

Determine the linearized and extended Kalman estimator equations.

SOLUTION TO EXERCISE 5.2 (a) Linearized Kalman Filter:

$$
\begin{aligned}
F^{[1]} &= \left.\frac{\partial}{\partial x} f(x(t),\, t) \right|_{x(t)=x^{\mathrm{NOM}}} \\
&= -x^{\mathrm{NOM}} \;=\; -1 \\
H^{[1]} &= \left.\frac{\partial}{\partial x(t)} h(x(t),\, t) \right|_{x(t)=\mathrm{NOM}} \\
&= 3(x^{\mathrm{NOM}})^2 = 3 \\
\dot{P}(t) &= F^{[1]}(t) \;\; P(t) + \;\; P(t)\; F^{[1]T}(t) + G(t)\; Q(t)\; G^T(t) - \bar{k}(t)\; R(t)\; \bar{k}^T(t) \\
\dot{P}(t) &= -1 * P(t) - 1 * P(t) + 1 - 0.5(\bar{k})^2 \\
&= -2P(t) + 1 - 0.5\bar{k}^{-2}(t) && (5.102) \\
\bar{k}(t) &= 6\, P(t) && (5.103) \\
\hat{\delta x}(t) &= -\hat{\delta x}_(t) + \bar{k}(t)[Z(t) + 3\; \hat{\delta x}(t)] && (5.104) \\
\hat{x}(t) &= \hat{\delta x}(t) + 1 && (5.105)
\end{aligned}
$$

(b) Extended Kalman Filter:

$$
F^{[1]}(t) \;=\; \left.\frac{\partial}{\partial x} f(x,\, t) \right|_{x=\hat{x}(t)}
$$

$$\begin{aligned}
&= -\hat{x}(t) \\
H^{[1]}(t) &= \frac{\partial}{\partial x} h(x,\, t)|_{x=\hat{x}(t)} \\
&= 3(\hat{x}(t))^2 \\
\hat{x}(t) &= -0.5(\hat{x})^2 + \bar{k}(t) \;\; [Z(t) - (\hat{x}(t))^3] &\quad (5.106) \\
\dot{P}(t) &= -\hat{x}(t)\; P(t) - \hat{x}(t)\; P(t) + 1 - 0.5(\bar{k}(t))^2 \\
&= -2\,\hat{x}(t)\; P(t) + 1 - 0.5\bar{k}^2(t) &\quad (5.107) \\
\bar{k}(t) &= 3\;(\hat{x}(t))^2\;\; P(t) \;\; / \;\; 0.5 &\quad (5.108) \\
&= 6\;(\hat{x}(t))^2\;\; P(t) &\quad (5.109)
\end{aligned}$$

5.5 Given the plant and measurement models for a scalar dynamic system:

$$\begin{aligned}
\dot{x}(t) &= ax(t) + w(t) \\
z(t) &= x(t) + v(t) \\
w(t) &\sim \mathcal{N}(0,1) \\
v(t) &\sim \mathcal{N}(0,2) \\
\mathrm{E}\,\langle x(0)\rangle &= 1 \\
\mathrm{E}\,\langle w(t)v(t)\rangle &= 0 \\
P(0) &= 2,
\end{aligned}$$

with unknown constant parameter a, derive an estimator for a, given $z(t)$.

SOLUTION TO EXERCISE 5.5 The nonlinear model is

$$\begin{aligned}
\dot{x}(t) &= a\,x(t) + w(t) \sim N(0,1) \qquad (1) \\
Z(t) &= x(t) + v(t) \sim N(0,2) \\
\dot{x}(t) &= a\,x(t) + w(t) \\
\dot{a}(t) &= 0.
\end{aligned}$$

The associated augmented system modle is:

$$\begin{aligned}
\dot{x}^*(t) &= \begin{bmatrix} \dot{x}(t) \\ \dot{a}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} a\,x(t) \\ 0 \end{bmatrix}}_{f(x^*,t)} + \underbrace{\begin{bmatrix} w(t) \\ 0 \end{bmatrix}}_{w^*(t)} \\
z(t) &= [1 \quad 0] \begin{bmatrix} x \\ a \end{bmatrix} + \overline{v(t)} \sim N(0,2) \\
Q^* &= Ew^*w^{*T} = \begin{bmatrix} E\,w(t)w^T(t) & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.
\end{aligned}$$

Therefore,

$$w^*(t) \sim \mathcal{N}\left(0, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)$$

and the extended Kalman filter model is

$$\begin{aligned}
F^{[1]}(t) &= \left.\frac{\partial f(x^*,t)}{\partial x^*}\right|_{x^*=\hat{x}^*(t)} = \begin{bmatrix} \hat{a}(t) & 0 \\ 0 & 0 \end{bmatrix} \\
H^{[1]}(t) &= \left.\frac{\partial h(x^*,t)}{\partial x^*}\right|_{x^*=\hat{x}^*(t)} = [1 \quad 0]
\end{aligned}$$