Chapter 5 Nonlinear Filtering Apr.-4,97 (10)

It is based on linearization of nonlinear systems

LINEARIZATION OF NONLINEAR SYSTEMS

(from GAJIC and LELIC, Prentice Hall, 1996

Consider now the general nonlinear dynamic control system in matrix form represented by

$$\frac{d}{dt}\mathbf{x}(t) = \mathcal{F}(\mathbf{x}(t), \mathbf{u}(t)) \tag{1.66}$$

where $\mathbf{x}(t)$, $\mathbf{u}(t)$, and $\mathcal{F}$ are, respectively, the n-dimensional state space vector, the r-dimensional control vector, and the n-dimensional vector function. Assume that the nominal (operating) system trajectory $\mathbf{x}_n(t)$ is known and that the nominal control that keeps the system on the nominal trajectory is given by $\mathbf{u}_n(t)$. Using the same logic as for the scalar case, we can assume that the actual system dynamics in the immediate proximity of the system nominal trajectories can be approximated by the first terms of the Taylor series. That is, starting with

$$\mathbf{x}(t) = \mathbf{x}_n(t) + \Delta\mathbf{x}(t), \qquad \mathbf{u}(t) = \mathbf{u}_n(t) + \Delta\mathbf{u}(t) \tag{1.67}$$

and

$$\frac{d}{dt}\mathbf{x}_n(t) = \mathcal{F}(\mathbf{x}_n(t), \mathbf{u}_n(t)) \tag{1.68}$$

we expand equation (1.66) as follows

$$\begin{aligned}\frac{d}{dt}\mathbf{x}_n + \frac{d}{dt}\Delta\mathbf{x} &= \mathcal{F}(\mathbf{x}_n + \Delta\mathbf{x}, \mathbf{u}_n + \Delta\mathbf{u}) \\ &= \mathcal{F}(\mathbf{x}_n, \mathbf{u}_n) + \left(\frac{\partial \mathcal{F}}{\partial \mathbf{x}}\right)_{\big|\substack{\mathbf{x}_n(t)\\ \mathbf{u}_n(t)}} \Delta\mathbf{x} + \left(\frac{\partial \mathcal{F}}{\partial \mathbf{u}}\right)_{\big|\substack{\mathbf{x}_n(t)\\ \mathbf{u}_n(t)}} \Delta\mathbf{u} + \text{high–order terms}\end{aligned} \tag{1.69}$$

High-order terms contain at least quadratic quantities of $\Delta\mathbf{x}$ and $\Delta\mathbf{u}$. Since $\Delta\mathbf{x}$ and $\Delta\mathbf{u}$ are small their squares are even smaller, and hence the high-order terms can be neglected. Using (1.67) and neglecting high-order terms, an approximation is obtained

$$\frac{d}{dt}\Delta\mathbf{x}(t) = \left(\frac{\partial \mathcal{F}}{\partial \mathbf{x}}\right)_{\big|\substack{\mathbf{x}_n(t)\\ \mathbf{u}_n(t)}} \Delta\mathbf{x}(t) + \left(\frac{\partial \mathcal{F}}{\partial \mathbf{u}}\right)_{\big|\substack{\mathbf{x}_n(t)\\ \mathbf{u}_n(t)}} \Delta\mathbf{u}(t) \tag{1.70}$$

(2)

Partial derivatives in (1.70) represent the Jacobian matrices given by

$$\left(\frac{\partial \mathcal{F}}{\partial \mathbf{x}}\right)\Big|_{\substack{\mathbf{x}_n(t)\\ \mathbf{u}_n(t)}} = \mathbf{A}^{n \times n} = \begin{bmatrix} \frac{\partial \mathcal{F}_1}{\partial x_1} & \frac{\partial \mathcal{F}_1}{\partial x_2} & \cdots & \cdots & \frac{\partial \mathcal{F}_1}{\partial x_n} \\ \frac{\partial \mathcal{F}_2}{\partial x_1} & \cdots & \cdots & \cdots & \frac{\partial \mathcal{F}_2}{\partial x_n} \\ \cdots & \cdots & \frac{\partial \mathcal{F}_i}{\partial x_j} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \mathcal{F}_n}{\partial x_1} & \frac{\partial \mathcal{F}_n}{\partial x_2} & \cdots & \cdots & \frac{\partial \mathcal{F}_n}{\partial x_n} \end{bmatrix}\Bigg|_{\substack{\mathbf{x}_n(t)\\ \mathbf{u}_n(t)}} \tag{1.71a}$$

$$\left(\frac{\partial \mathcal{F}}{\partial \mathbf{u}}\right)\Big|_{\substack{\mathbf{x}_n(t)\\ \mathbf{u}_n(t)}} = \mathbf{B}^{n \times r} = \begin{bmatrix} \frac{\partial \mathcal{F}_1}{\partial u_1} & \frac{\partial \mathcal{F}_1}{\partial u_2} & \cdots & \cdots & \frac{\partial \mathcal{F}_1}{\partial u_r} \\ \frac{\partial \mathcal{F}_2}{\partial u_1} & \cdots & \cdots & \cdots & \frac{\partial \mathcal{F}_2}{\partial u_r} \\ \cdots & \cdots & \frac{\partial \mathcal{F}_i}{\partial u_j} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \mathcal{F}_n}{\partial u_1} & \frac{\partial \mathcal{F}_n}{\partial u_2} & \cdots & \cdots & \frac{\partial \mathcal{F}_n}{\partial u_r} \end{bmatrix}\Bigg|_{\substack{\mathbf{x}_n(t)\\ \mathbf{u}_n(t)}} \tag{1.71b}$$

Note that the Jacobian matrices have to be evaluated at the nominal points, i.e. at $\mathbf{x}_n(t)$ and $\mathbf{u}_n(t)$. With this notation, the linearized system (1.70) has the form

$$\frac{d}{dt}\Delta\mathbf{x}(t) = \mathbf{A}\Delta\mathbf{x}(t) + \mathbf{B}\Delta\mathbf{u}(t), \quad \Delta\mathbf{x}(t_0) = \mathbf{x}(t_0) - \mathbf{x}_n(t_0) \tag{1.72}$$

The output of a nonlinear system, in general, satisfies a nonlinear algebraic equation, that is

$$\mathbf{y}(t) = \mathcal{G}(\mathbf{x}(t), \mathbf{u}(t)) \tag{1.73}$$

This equation can be also linearized by expanding its right-hand side into a Taylor series about nominal points $\mathbf{x}_n(t)$ and $\mathbf{u}_n(t)$. This leads to

$$\mathbf{y}_n + \Delta\mathbf{y} = \mathcal{G}(\mathbf{x}_n, \mathbf{u}_n) + \left(\frac{\partial \mathcal{G}}{\partial \mathbf{x}}\right)\Big|_{\substack{\mathbf{x}_n(t)\\ \mathbf{u}_n(t)}} \Delta\mathbf{x} + \left(\frac{\partial \mathcal{G}}{\partial \mathbf{u}}\right)\Big|_{\substack{\mathbf{x}_n(t)\\ \mathbf{u}_n(t)}} \Delta\mathbf{u} \\ + \text{high–order terms} \tag{1.74}$$

Note that $\mathbf{y}_n$ cancels term $\mathcal{G}(\mathbf{x}_n, \mathbf{y}_n)$. By neglecting high-order terms in (1.74), the linearized part of the output equation is given by

$$\Delta\mathbf{y}(t) = \mathbf{C}\Delta\mathbf{x}(t) + \mathbf{D}\Delta\mathbf{u}(t) \tag{1.75}$$

where the Jacobian matrices $\mathbf{C}$ and $\mathbf{D}$ satisfy

$$\mathbf{C}^{p\times n} = \left(\frac{\partial \mathcal{G}}{\partial \mathbf{x}}\right)\Big|_{\substack{\mathbf{x}_n(t)\\ \mathbf{u}_n(t)}} = \begin{bmatrix} \frac{\partial \mathcal{G}_1}{\partial x_1} & \frac{\partial \mathcal{G}_1}{\partial x_2} & \cdots & \cdots & \frac{\partial \mathcal{G}_1}{\partial x_n} \\ \frac{\partial \mathcal{G}_2}{\partial x_1} & \cdots & \cdots & \cdots & \frac{\partial \mathcal{G}_2}{\partial x_n} \\ \cdots & \cdots & \frac{\partial \mathcal{G}_i}{\partial x_j} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \mathcal{G}_p}{\partial x_1} & \frac{\partial \mathcal{G}_p}{\partial x_2} & \cdots & \cdots & \frac{\partial \mathcal{G}_p}{\partial x_n} \end{bmatrix}\Bigg|_{\substack{\mathbf{x}_n(t)\\ \mathbf{u}_n(t)}} \tag{1.76a}$$

$$\mathbf{D}^{p\times r} = \left(\frac{\partial \mathcal{G}}{\partial \mathbf{u}}\right)\Big|_{\substack{\mathbf{x}_n(t)\\ \mathbf{u}_n(t)}} = \begin{bmatrix} \frac{\partial \mathcal{G}_1}{\partial u_1} & \frac{\partial \mathcal{G}_1}{\partial u_2} & \cdots & \cdots & \frac{\partial \mathcal{G}_1}{\partial u_r} \\ \frac{\partial \mathcal{G}_2}{\partial u_1} & \cdots & \cdots & \cdots & \frac{\partial \mathcal{G}_2}{\partial u_r} \\ \cdots & \cdots & \frac{\partial \mathcal{G}_i}{\partial u_j} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \mathcal{G}_p}{\partial u_1} & \frac{\partial \mathcal{G}_p}{\partial u_2} & \cdots & \cdots & \frac{\partial \mathcal{G}_p}{\partial u_r} \end{bmatrix}\Bigg|_{\substack{\mathbf{x}_n(t)\\ \mathbf{u}_n(t)}} \tag{1.76b}$$

Example 1.2: Let a nonlinear system be represented by

$$\frac{dx_1}{dt} = x_1 \sin x_2 + x_2 u$$

$$\frac{dx_2}{dt} = x_1 e^{-x_2} + u^2$$

$$y = 2x_1 x_2 + x_2^2$$

Assume that the values for the system nominal trajectories and control are known and given by x_{1n}, x_{2n}, and u_n. The linearized state space equation of the above nonlinear system is obtained as

$$\begin{bmatrix} \Delta\dot{x}_1(t) \\ \Delta\dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \sin x_{2n} & x_{1n}\cos x_{2n} + u_n \\ e^{-x_{2n}} & -x_{1n}e^{-x_{2n}} \end{bmatrix} \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \end{bmatrix} + \begin{bmatrix} x_{2n} \\ 2u_n \end{bmatrix} \Delta u(t)$$

$$\Delta y(t) = \begin{bmatrix} 2x_{2n} & 2x_{1n} + 2x_{2n} \end{bmatrix} \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \end{bmatrix} + 0\Delta u(t)$$

Having obtained the solution of this linearized system under the given control input $\Delta u(t)$, the corresponding approximation of the nonlinear system trajectories is

$$\mathbf{x}_n(t) + \Delta\mathbf{x}(t) = \begin{bmatrix} x_{1n}(t) \\ x_{2n}(t) \end{bmatrix} + \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \end{bmatrix}$$

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(3)

Example 1.3: Consider the mathematical model of a single-link robotic manipulator with a flexible joint (Spong and Vidyasagar, 1989)

$$I\ddot{\theta}_1 + mgl\sin\theta_1 + k(\theta_1 - \theta_2) = 0$$
$$J\ddot{\theta}_2 - k(\theta_1 - \theta_2) = u$$

where θ_1, θ_2 are angular positions, I, J are moments of inertia, m and l are, respectively, the link's mass and length, and k is the link's spring constant. Introducing the change of variables as

$$x_1 = \theta_1,\ x_2 = \dot{\theta}_1,\ x_3 = \theta_2,\ x_4 = \dot{\theta}_2$$

the manipulator's state space nonlinear model equivalent to (1.66) is given by

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{mgl}{I}\sin x_1 - \frac{k}{I}(x_1 - x_3) \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \frac{k}{J}(x_1 - x_3) + \frac{1}{J}u
\end{aligned}$$

Take the nominal points as $(x_{1n}, x_{2n}, x_{3n}, x_{4n}, u_n)$, then the matrices $\mathbf{A}$ and $\mathbf{B}$ defined in (1.71) are given by

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k + mgl\cos x_{1n}}{I} & 0 & \frac{k}{I} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{J} & 0 & -\frac{k}{J} & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{J} \end{bmatrix}$$

In Spong (1995) the following numerical values are used for system parameters:- $mgl = 5,\ I = J = 1,\ k = 0.08$.

Assuming that the output variable is equal to the link's angular position, that is

$$y = x_1$$

the matrices $\mathbf{C}$ and $\mathbf{D}$, defined in (1.76), are given by

$$\mathbf{C} = [1 \quad 0 \quad 0 \quad 0], \quad \mathbf{D} = 0$$

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5.1 Chapter Focus

5.2 Problem Statement

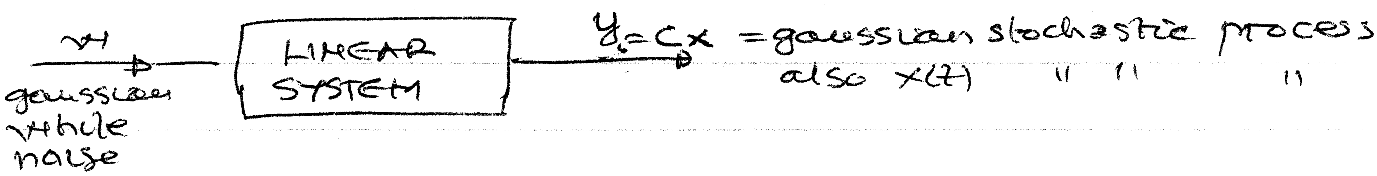
Nonlinear Plant and Measurement Models

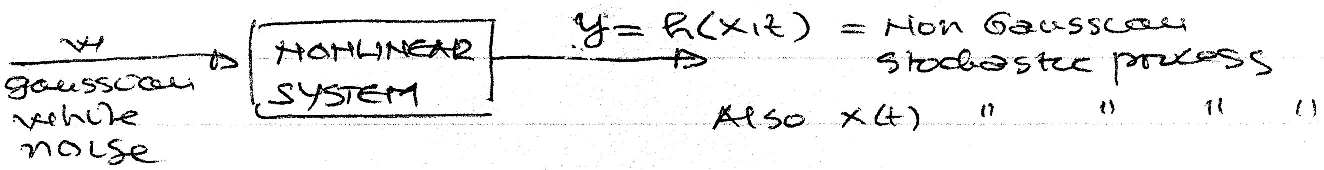
<u>Continuous time</u>

$$\dot{x} = f(x,t) + G w, \qquad E\{w(t)\}=0, \; E\{w(t)w^T(\tau)\}=Q(t)\delta(t-\tau)$$

$$z = h(x,t) + v, \qquad E\{v(t)\}=0, \; E\{v(t)v^T(\tau)\}=R(t)\delta(t-\tau)$$

It is not necessary that w and v are Gaussian. Why





<u>Discrete time</u>

$$x_k = f(x_{k-1}, k-1) + w_{k-1}$$

$$z_k = h(x_k, k) + v_k$$

$$E\{w_k\} = 0$$

$$E\{w_k w_i^T\} = Q_k \Delta(k-i)$$

$$E\{v_k\} = 0$$

$$E\{v_k v_i^T\} = R_k \Delta(k-i)$$

$$E\{x_0\} = \text{known}$$

$$Cov\{x_0\} = \text{known}$$

(5.3) f and h are twice continuously differentiable.
$\Rightarrow$ linearization possible

(5.4) LINEARIZATION OF DISCRETE-TIME NONLINEAR SYSTEMS ABOUT A NOMINAL TRAJECTORY

(1) $$x_k = f(x_{k-1}, k-1) + w_{k-1}$$

$$x_k^{NOM} = f(x_{k-1}^{NOM}, k-1) \qquad x_0^{NOM} = \text{given}$$

obtained when the random variables take their expected values.

Let

$$\delta x_k = x_k - x_k^{NOM}$$

$$\delta z_k = z_k - h(x_k^{NOM}, k)$$

Expand $f(x_{k-1}, k-1)$ into the Taylor series at x_{k-1}^{NOM}

$$f(x_{k-1}, k-1) = f(x_{k-1}^{NOM}, k-1) + \left.\frac{\partial f(x, k-1)}{\partial x}\right|_{x = x_{k-1}^{NOM}} \delta x_{k-1} + \cancelto{0}{h.o.t.}$$

$$x_k = \delta x_k + x_k^{NOM} = f(x_{k-1}, k-1) + w_{k-1}$$

$$\delta x_k + \cancel{x_k^{NOM}} \approx \cancel{f(x_{k-1}^{NOM}, k-1)} + \left.\frac{\partial f(x, k-1)}{\partial x}\right|_{x = x_{k-1}^{NOM}} \delta x_{k-1} + w_{k-1}$$

$$\delta x_k \approx \left.\frac{\partial f(x, k-1)}{\partial x}\right|_{x = x_{k-1}^{NOM}} \cdot \delta x_{k-1} + w_{k-1} = \Phi_k \delta x_{k-1} + w_{k-1}$$

thus

(2) $$\boxed{\delta x_k = \Phi_k \delta x_{k-1} + w_{k-1}}$$

is the linearized model of (1).

Φ_k is the Jacobian matrix

$$\Phi_k = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & & & \\ \frac{\partial f_n}{\partial x_1} & \cdots & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \Bigg|_{x = x_{k-1}^{NOM}}$$

LINEARIZATION OF THE OUTPUT EQUATION

(3) $\boxed{z_k = h(x_k, k) + v_k}$, $\dim z_k = \ell$

$$z_k^{NOM} = h(x_k^{NOM}, k)$$

$$h(x_k, k) = h(x_k^{NOM}, k) + \frac{\partial h(x,k)}{\partial x}\Big|_{x = x_k^{NOM}} \delta x_k + \underbrace{\cancel{h.o.t.}}_{\to 0}$$

$$\delta z_k = z_k - h(x_k^{NOM}, k)$$

$$\delta z_k + \cancel{h(x_k^{NOM}, k)} = z_k = h(x_k, k) + v_k$$

$$\approx \cancel{h(x_k^{NOM}, k)} + \frac{\partial h(x,k)}{\partial x}\Big|_{x = x_k^{NOM}} \delta x_k + v_k$$

thus

$$\delta z_k \approx \frac{\partial h(x,k)}{\partial x}\Big|_{x = x_k^{NOM}} \delta x_k + v_k = H_k \delta x_k + v_k$$

(4) $\boxed{\delta z_k = H_k \delta x_k + v_k}$

$$H_k = \frac{\partial h(x,k)}{\partial x}\Big|_{x = x_k^{NOM}} = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \frac{\partial h_2}{\partial x_1} & & & \\ \vdots & & & \\ \frac{\partial h_\ell}{\partial x_1} & \cdots & \cdots & \frac{\partial h_\ell}{\partial x_n} \end{bmatrix}$$

Continuous-time case:

$$\dot{x} = f(x,t) + G w(t)$$
$$z = h(x,t) + v(t)$$

$$x = x_n + \delta x$$

$$\Rightarrow \quad \delta\dot{x} = \left\{ \frac{\partial f}{\partial x}\bigg|_{x=x^{NOM}} \right\} \delta x + G w(t) = F\delta x + G w$$

Also

$$\delta z = \left\{ \frac{\partial h}{\partial x}\bigg|_{x=x^{NOM}} \right\} \cdot \delta x + v(t) = H\delta x + v$$

5.5 LINEARIZATION ABOUT THE ESTIMATED TRAJECTORY

The nominal trajectory is replaced by the estimated trajectory. The rest is the same.

$$\Rightarrow \quad \left. \begin{aligned} \Phi(\hat{x},k) &= \frac{\partial f(x,k)}{\partial x}\bigg|_{x=\hat{x}_k(-)} \\ H(\hat{x},k) &= \frac{\partial h(x,k)}{\partial x}\bigg|_{x=\hat{x}_k(-)} \end{aligned} \right\} \Rightarrow \begin{aligned} \delta x_k &= \Phi(\hat{x},k-1)\delta x_{k-1} + w_{k-1} \\ \delta z_k &= H(\hat{x},k)\delta x_k + v_k \end{aligned}$$

or in the continuous-time domain

$$\left. \begin{aligned} F(t) &= \frac{\partial f(x,t)}{\partial x}\bigg|_{x=\hat{x}(t)} \\ H(t) &= \frac{\partial h(x,t)}{\partial x}\bigg|_{x=\hat{x}(t)} \end{aligned} \right\} \Rightarrow \begin{aligned} \delta\dot{x} &= F\delta x + G w \\ \delta z &= H\delta x + v \end{aligned}$$