330: 519 Advanced Topics in Systems Engineering, Spring 1997

Kalman Filtering

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Text-book: M. Grewal and A. Andrews, *Kalman Filtering: Theory and Practice*, Prentice Hall, Englewood Cliffs, 1993

Office Hours: After the class and by appointment

Class home page: http://www.ece.rutgers.edu/~gajic/519.html

Topics:

Week 1: Introduction to Kalman Filtering and Linear Dynamic Systems (Chapters 1 and 2)

Week 2: Review of Probability and Random processes (Sections 3.1–3.4)

Week 3: Linear Stochastic Systems, Shaping Filters, Derivations of the Covariance Equation in Continuous- and Discrete-Time Domains, and Orthogonality Principle (Section 3.5–3.8)

Week 4: Discrete- and Continuous-Time Kalman Filter (Sections 4.1–4.7)

Week 5: Continuous- and Discrete-Time Riccati Equations (Sections 4.8–4.10)

Week 6: Applications of Kalman Filters, Smoothers, Examples (Sections 4.11–4.13)

Week 7: Discussion of Main Journal Papers on Kalman Filtering

Week 8: Applications of Kalman Filter in Signal Processing and Communications (*IEEE Transactions* Journal Papers). Project Assignments

Week 9: Exam I

Week 10: Extended Continuous- and Discrete-Time Kalman Filters (Chapter 5)

Week 11: Implementation Methods (Sections 6.1–6.4)

Week 12: Implementation Methods (Sections 6.5–6.8)

Week 13: Practical Considerations (Chapter 7)

Week 14: Project presentations

Grading:
Exam I 30%
Project 30%
Final Exam 40%

Kalman Filtering

Chapter I

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1.1 On Kalman Filtering

1.2 On Estimation Methods

1.2.2 Method of **Least Squares**

Example 1.1 Solution for Overdetermined Systems of Linear Algebraic Equations

$$Hx = z \qquad H \in \mathbb{R}^{m\times n}, \quad m > n$$

$$H = \{h_{ij}\}, \; i = 1,\dots,n, \; j = 1,\dots,m \qquad x \in \mathbb{R}^n, \; z \in \mathbb{R}^m$$

more equations than unknowns

$\Rightarrow$ no exact solution, but it is possible to get the so-called least square solution, $\hat{x}$,

$$\min_{\hat{x}} e^2 = \min_{\hat{x}} \|H\hat{x} - z\|_2^2 = \min_{\substack{\hat{x}_k \\ k=1,\dots,n}} \sum_{i=1}^{m} \Big[\sum_{j=1}^{n} h_{ij}\hat{x}_j - z_i\Big]^2$$

$\| \; \|_2$ = second norm = Euclidean vector norm

$\|v\|_2^2 = \langle v, v\rangle = v^T v$

$$\frac{\partial e^2}{\partial \hat{x}_k} = 0 = 2\sum_{i=1}^{m} h_{ik}\Big[\sum_{j=1}^{n} h_{ij}\hat{x}_j - z_i\Big]$$

$k = 1, 2, \dots, n$

or in matrix form

$$0 = 2H^T(H\hat{x} - z)$$

$$0 = \cancel{2}H^T H\hat{x} - \cancel{2}H^T z$$

$\Rightarrow$

$$H^T H\hat{x} = H^T z$$ known as the normal equation

If $\det H^T H \neq 0 \Leftrightarrow \mathrm{rank}(H^T H) = n \Rightarrow$

$$\boxed{\hat{x} = (H^T H)^{-1} H^T z}$$

$H^T H$ = Grammian matrix

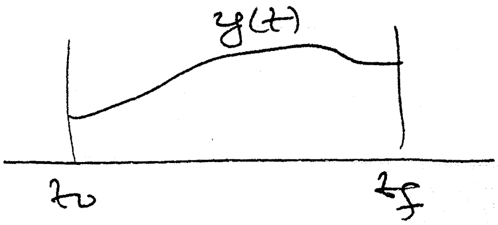
Matrix notation

$$\min_{\hat{x}} e^2 = \min_{\hat{x}} \| H\hat{x} - z \|_2^2 = \min_{\hat{x}} \langle H\hat{x} - z, H\hat{x} - z \rangle$$

$$= \min_{\hat{x}} \left\{ (\hat{x}^T H^T - z^T)(H\hat{x} - z) \right\} = \min_{\hat{x}} \left\{ \hat{x}^T H^T H \hat{x} - \hat{x}^T H^T z - z^T H \hat{x} + z^T z \right\}$$

$$\Rightarrow 2 H^T H \hat{x} - H^T z - H^T z = 0 \Rightarrow \boxed{H^T H \hat{x} = H^T z}$$

Example 1.2 Continuous-Time Least Squares



$y(t) = A(t)\, x$, x = unknown vector

$y \in \mathbb{R}^e$, $A^{e \times n}$, $x \in \mathbb{R}^n$

$$\| e(t) \|_2^2 = \| y(t) - A(t) x \|_2^2 = \langle (y(t) - A(t)x), (y(t) - A(t)x) \rangle$$

$$= \left(y^T(t) - x^T A^T(t) \right) \left(y(t) - A(t) x \right)$$

$$= y^T(t) y(t) - y^T(t) A(t) x - x^T A^T(t) y(t) + x^T A^T(t) A(t) x$$

Criterion is to minimize $\int_{t_0}^{t_f} \| e(t) \|_2^2 \, dt$

$$\min_x \left\{ \int_{t_0}^{t_f} y^T(t) y(t)\, dt + x^T \int_{t_0}^{t_f} A^T(t) A(t)\, dt\, x - 2 x^T \left(\int_{t_0}^{t_f} A^T(t) y(t)\, dt \right) \right\}$$

$$2 \int_{t_0}^{t_f} A^T(t) A(t)\, dt\, \hat{x} - 2 \left(\int_{t_0}^{t_f} A^T(t) y(t)\, dt \right) = 0$$

$$\Rightarrow \hat{x} = \underbrace{\left(\int_{t_0}^{t_f} A^T(t) A(t)\, dt \right)}_{\substack{\text{Gramian matrix} \\ \text{must be nonsingular} \\ \text{for } \forall t \in [t_0, t_f]}}^{-1} \left(\int_{t_0}^{t_f} A^T(t) y(t)\, dt \right)$$

1.2.3 In this book
Observable means also uniquely determinable

1.2.5 Wiener Filter (based on the autocorrelation function of the signal and the noise

1.2.6 Kalman Filter (1960)
Its steady state version = Wiener Filter (194
also known as Wiener-Kolmogorov Filter

1.2.7 Numerical stability and efficiency

1.2.8 Extended Kalman filter
= Kalman filter for non-linear systems.

1.3 Notation

1.4 Chapter Summary

Important material from chapter I, pages 7-9.
The rest, read only once.

Chapter II - Intro to Dynamic Systems

(2-1)

$$\dot{x}(t) = F(t)x(t) + C(t)u(t), \quad x(t_0) = x_0, \quad x \in \mathbb{R}^n$$

State space model of a linear continuous time dynamic system

$F(t), C(t) \Rightarrow$ time varying

$F = \text{const}, C = \text{const} \Rightarrow$ time invariant

$\dot{x}(t) = f(x(t), u(t))$ = nonlinear continuous-time system

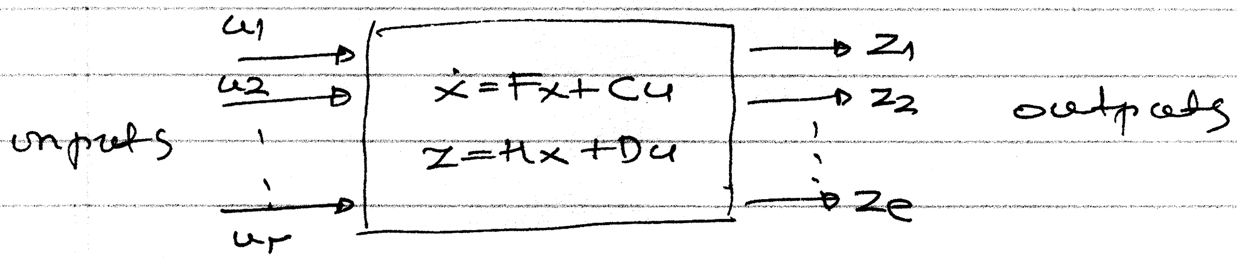
$$x(t_{k+1}) = \Phi(t_k)x(t_k) + \Gamma(t_k)u(t_k)$$

State space model of a linear discrete-time dynamic system

$x(t_{k+1}) = \varphi(x(t_k), u(t_k))$ = nonlinear discrete-time system

2.2) Dynamic Systems

2.3) Continuous Linear Systems



$$F \in \mathbb{R}^{n \times n}, \quad C \in \mathbb{R}^{n \times r}, \quad H \in \mathbb{R}^{e \times n}, \quad D \in \mathbb{R}^{e \times r}$$

2.3.3 Companion Form

$$\frac{d^n y}{dt^n} + f_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + f_{n-1}\frac{dy}{dt} + f_n y = u$$

$x_1 = y \Rightarrow \dot{x}_1 = \dot{y} = x_2$

$x_2 = \dot{y} \qquad \dot{x}_2 = \ddot{y} = x_3$

$x_3 = \ddot{y}$

$\vdots$

$x_n = y^{n-1} \qquad \dot{x}_n = y^{(n)} = -f_1 y^{(n-1)} - \cdots - f_{n-1}\frac{dy}{dt} - f_n y + u$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 \cdots & 0 \\ & & & \ddots & \\ & & & & 1 \\ -f_n & -f_{n-1} & \cdots & & -f_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

nice and simple form, but not recommended for implementation (it is not robust for numerical computations ⇒ ill-conditioning)

$$y = [1 \ 0 \ \cdots \ 0]\, x$$

$$z(t) = H(t)\, x(t) + D(t)\, u(t)$$

input/output coupling matrix

measurement vector

measurement sensitivity matrix

2.3.6 Solving Differential Equations

$$\dot{x}(t) = F(t)\, x(t)$$

$\Phi(t)$ = state transition matrix

$$x(t) = \Phi(t, t_0)\, x(t_0)$$

Note that

$$\frac{d}{dt}\Phi(t, t_0) = A(t)\, \Phi(t, t_0), \qquad \Phi(t_0, t_0) = I$$

(page 36, $\Phi(t)$ = not precisely defined)

$\phi(t, t_0)$ is nonsingular for every t.

$\psi(t)$ = system fundamental matrix

$$\phi(t, t_0) = \psi(t)\, \psi^{-1}(t_0)$$

$\psi(t)$ satisfies the same equation

$$\dot{\psi}(t) = A(t)\, \psi(t)$$

but without initial conditions imposed. In addition $\det \psi(t) \neq 0$ that is, its columns must be linearly independent

Some properties of $\phi(t, t_0)$

1) $\phi(t_0, t_0) = I$
2) $\phi^{-1}(t, t_0) = \phi(t_0, t)$
3) $\phi(t_2, t_1)\, \phi(t_1, t_0) = \phi(t_2, t_0)$
4) $\dfrac{\partial \phi(t, t_0)}{\partial t} = F(t)\, \phi(t, t_0)$
5) $\dfrac{\partial \phi(t, t_0)}{\partial t_0} = -\Phi(t, t_0)\, F(t)$

$\boxed{2.3.7}$ Solution of Nonhomogeneous Equation

$$x(t) = \phi(t, t_0)\, x(t_0) + \int_{t_0}^{t} \phi(t, \tau)\, C(\tau)\, u(\tau)\, d\tau$$

for time invariant systems

$$\phi(t, t_0) = e^{A(t-t_0)}$$

$$x(t) = e^{A(t-t_0)} x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}\, C(\tau)\, d\tau$$

There are 19 ways to calculate e^{At}.

Classroom methods

1) $e^{At} = I + \frac{At}{1!} + \frac{A^2t^2}{2!} + \dots$

2) $e^{At} = \mathcal{L}^{-1}((sI-A)^{-1})$ (Laplace method)

3) Cayley-Hamilton method

(2.4) DISCRETE-TIME SYSTEMS

$$x_k = \phi_{k-1} x_{k-1} + \Gamma_{k-1} u_{k-1}$$

$$z_k = H_k x_k + D_k u_k$$

$$\Longrightarrow \quad x_k = \phi^k x_0 + \sum_{i=0}^{k-1} \phi^{k-i-1} \Gamma u_i$$

Φ^k = discrete-time state transition matrix (more precisely $\phi^{k-k_0} = \phi^{k-0} = \phi^k$)

It can be evaluated by using the Z-transform method

$$\phi^k = \mathcal{Z}^{-1}[z(zI-\Phi)^{-1}]$$

(2-5) OBSERVABILITY

Observability means: the state of a dynamic system is uniquely determinable from its inputs and outputs

Observability Gramian

$$\textcircled{1}(H, F, t_0, t_f) = \int_{t_0}^{t_f} \Phi^T(t, t_0) H^T(t) H(t) \Phi(t, t_0)\, dt$$

for time invariant systems

$$\text{rank}\left[H^T \; F^T H^T \; (F^T)^2 H^T \; - - (F^T)^{n-1} H^T\right] = n$$

$\Rightarrow$ observability

(2.5.2) Controllability;

$$\dot{x} = Fx + Cu, \quad x(t_0) = x_0$$
$$z = Hx + Du$$

The system is controllable at time $t = t_0$ iff for any $x(t_f)$, $t_f < \infty$, there exists a piecewise continuous input function $u(t)$ which drives the system from $x(t_0)$ to $x(t_f)$.

Rank test;

$$\text{rank}\ S = [C \; FC \; F^2C \; - - F^{n-1}C] = n$$

~~2.6~~