8.7.1 Experiment 1—The Inverted Pendulum

Part 1. The linearized equations of the inverted pendulum, obtained by assuming that the pendulum mass is concentrated at its center of gravity [15, 16] are given by

$$\begin{aligned} \left(J + mL^2\right)\ddot{\theta}(t) - mgL\theta(t) + mL\ddot{d}(t) = 0 \\ (M + m)\ddot{d}(t) + mL\ddot{\theta}(t) = f(t) \end{aligned} \tag{8.148}$$

where $\theta(t)$ is the angle of the pendulum from the vertical position, $d(t)$ is the position of the cart, $f(t)$ is the force applied to the cart, M is the mass of the cart, m is the mass of the pendulum, g is the gravitational constant, and J is the moment of inertia about the center of mass. Assuming that normalized values are given by $J = 1$, $L = 1$, $g = 9.81$, $M = 1$, and $m = 0.1$, derive the state space form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}f(t)$$

where

$$\mathbf{x}(t) = [\theta(t) \quad \dot{\theta}(t) \quad d(t) \quad \dot{d}(t)]^T$$

and $\mathbf{A}^{4\times 4}$ and $\mathbf{B}^{4\times 1}$ are the corresponding matrices.

Part 2. Using MATLAB, determine the following:

(a) The eigenvalues, eigenvectors, and characteristic polynomial of matrix $\mathbf{A}$.

(b) The state transition matrix at the time instant $t = 1$.

(c) The unit impulse response (take $\theta(t)$ and $d(t)$ as the output variables) for $0 \le t \le 1$ with $\Delta t = 0.1$. Plot the system output response.

(d) The unit step response for $0 \le t \le 1$ and $\Delta t = 0.1$. Draw the system output response.

(e) The unit ramp response for $0 \le t \le 1$ and $\Delta t = 0.1$. Draw the system output response. Compare the response diagrams obtained in (c), (d), and (e).

(f) The system state response resulting from the initial condition $\mathbf{x}(0) = [-1 \quad 1 \quad 1 \quad 1]^T$ and the input $f(t) = \sin(t)$ for $0 \le t \le 5$ and $\Delta t = 0.1$.

(g) The inverse of the state transition matrix $\left(e^{\mathbf{A}t}\right)^{-1}$ for $t = 5$.

(h) The state $\mathbf{x}(t)$ at time $t = 5$ assuming that $\mathbf{x}(10) = [10 \quad 0 \quad 5 \quad 2]^T$ and $f(t) = 0$, using the result from (g).

(i) Find the system transfer function.

Part 3. Discretize the continuous-time system defined in (8.148) with $T = 0.02$, and find the discrete-time space model

$$\mathbf{x}[k+1] = \mathbf{A}_d\mathbf{x}[k] + \mathbf{B}_d f[k]$$

Assuming that the output equation of the discrete system is given by

$$\mathbf{y}[k] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x}[k] = \mathbf{C}_d \mathbf{x}[k]$$

find the system output response for $0 \leq k \leq 50$ due to initial conditions $\mathbf{x}_0 = [-1 \quad 1 \quad -1 \quad 1]^T$ and unit step input (note that $f[k]$ should be generated as a column vector of 51 elements equal to 1).

Part 4. Consider the continuous-time system given by

$$\frac{d^2 y(t)}{dt^2} + 0.1 \frac{dy(t)}{dt} = f(t) \tag{8.149}$$

(a) Discretize this system with $T = 1$ using the Euler approximation.

(b) Find the system state and output responses of the obtained discrete system for $k = 1, 2, 3, ..., 20$, when $f(t) = \sin(0.1\pi t)$ and $y(0) = \dot{y}(0) = 0$.

(c) Find discrete transfer function, characteristic equation, eigenvalues, and eigenvectors.

Part 5. Discretize the state space form of (8.149) obtained using MATLAB function `c2d` with $T = 1$. Find the discrete system state and output responses for the initial condition and the input function defined in Part 4(b). Compare the results obtained in Parts 4 and 5. Comment on the results obtained.