MC-MIMO Radar:
Recoverability and Performance Bounds

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Abstract—It was recently shown that low rank matrix completion theory can be employed for designing new sampling schemes in the context of MIMO radars, which can lead to the reduction of the high volume of data typically required for accurate target detection and estimation. This paper focuses on the performance of matrix completion in colocated MIMO radar systems equipped with Uniform Linear Arrays (ULAs). Exploiting the particular structure of the received data matrix, we present novel theoretical results showing that its coherence is both asymptotically and approximately optimal with respect to the number of antennas of the arrays involved and further, that the data matrix is recoverable using a subset of its entries with minimal cardinality.

Index Terms—Matrix Completion, Subspace Coherence, Strong Incoherence Property, Colocated MIMO Radar, Array Processing

I. INTRODUCTION

Recently, matrix completion has been proposed as means for effectively reducing the volume of data required for target detection and estimation in MIMO radars [1] and more generally in array processing systems [2], [3], [4]. In [4], a colocated MIMO radar [5] scenario is considered, in which the transmission and reception antennas are organized in Uniform Linear Arrays (ULAs). Each transmission antenna transmits a narrowband waveform over a predefined carrier frequency, with the waveforms between different transmission antennas being orthogonal. At each reception antenna, after demodulation, matched filtering is performed with the transmit waveforms [4], extracting statistics and formulating a matrix (referred to here as the data matrix), which can be used by standard array processing methods for target detection and parameter estimation. For a sufficiently large number of transmission and reception antennas and a small number of targets, the data matrix is low-rank. Therefore, one would hope it can be recovered from a small number of its entries via matrix completion. This implies that, at each reception antenna, matched filtering does not need to be performed with all transmit waveforms, but rather with a small number of randomly selected ones from the waveform dictionary.

In this paper, we consider the matrix completion enabled MIMO radar system proposed in [4] and show that, due to the special structure of the data matrix, one can derive very insightful theoretical results regarding its coherence (see Section II). Our contribution is summarized as follows:

1) We show that, for ULA configurations, and under mild assumptions on the Directions-Of-Arrival (DOAs) of the targets, the coherence of the data matrix is both asymptotically optimal with respect to the number of transmission and reception antennas, and nearly optimal for a sufficiently large but finite number of transmission and reception antennas.

2) Under common assumptions regarding the range of the pairwise differences of the target angles and the spacing of the antennas of the ULAs involved, we derive a simple sufficient condition, which essentially controls the coherence of the data matrix, as well as the rate of convergence to its optimum value. In all cases, we provide explicit and computationally tractable coherence bounds, with all results holding almost surely.

3) Invoking the recent theoretical results in low rank matrix completion presented in [6], we derive asymptotic bounds on the number of observations required for exact matrix completion, showing that, in fact, the matrix under consideration can be reconstructed using a subset of its entries with minimal cardinality.

The paper is organized as follows. In Section II, we briefly introduce the required background in both noiseless and noisy matrix completion. We also restate the problem formulation for the ULA MIMO radar [4]. In Section III, we present our coherence and recoverability results. Finally, in Section IV, we validate the correctness of our previously stated results via numerical simulations.

II. BACKGROUND

A. Matrix Completion:
Problem Statement and Recoverability Conditions

Consider a generic matrix \( M \in \mathbb{C}^{N_1 \times N_2} \) of rank \( r \), whose compact Singular Value Decomposition (SVD) is given by \( M = U \Sigma V^H \), where \( \Sigma = \sum_{i \in \mathbb{N}_r^+} \sigma_i (M) u_i v_i^H \) and with column and row subspaces denoted as \( U \) and \( V \) respectively, spanned by the sets \( \{ u_i \in \mathbb{C}^{N_1 \times 1} \}_{i \in \mathbb{N}_r^+} \) and \( \{ v_i \in \mathbb{C}^{N_2 \times 1} \}_{i \in \mathbb{N}_r^+} \), respectively.

Let \( \mathcal{P}(M) \in \mathbb{C}^{N_1 \times N_2} \) denote an entrywise sampling of \( M \). In all the analysis that follows, we will adopt the theoretical frameworks presented in [7] and [6], according to which one hopes to reconstruct \( M \) from \( \mathcal{P}(M) \) by solving the convex

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program

\[
\text{minimize } \|X\|_2, \\
\text{subject to } X(i,j) = M(i,j), \quad \forall (i,j) \in \Omega,
\]

(1)

where the set \(\Omega\) contains all matrix coordinates corresponding to the observed entries of \(M\) (contained in \(\mathcal{P}(M)\)) and where \(\|X\|_2\) represents the nuclear norm of \(X\). In the following, we will refer to (1) as the Matrix Completion (MC) problem.

Also in [7], the authors introduce the notion of subspace coherence, in order to derive specific conditions under which the solution of (1) coincides with \(M\). The formal definition of subspace coherence follows, in a slightly more expanded form compared to the original definition stated in [7].

**Definition 1. (Subspace Coherence [7])** Let \(U \subseteq \mathbb{C}^r \subseteq \mathbb{C}^N\) be a subspace spanned by the set of orthonormal vectors \(\{u_i \in \mathbb{C}^{N \times 1}\}_{i \in \mathbb{N}_N^r}\). Also, define the matrix \(U = [u_1, u_2, \ldots, u_r] \in \mathbb{C}^{N \times r}\) and let \(P_U = UU^H \in \mathbb{R}^{N \times N}\) be the orthogonal projection onto \(U\). Then, the coherence of \(U\) with respect to the standard basis \(\{e_i\}_{i \in \mathbb{N}_N}\) is defined as

\[
\mu(U) \triangleq \frac{\sum_{i \in \mathbb{N}_N^r} \|u_i v_i^H\|_\infty}{\frac{r}{N_1 N_2}}, \quad \mu_1 \in \mathbb{R}_{++}.
\]

(2)

Concerning the recoverability of \(M\), the following assumptions regarding the subspaces \(U\) and \(V\) are of particular importance [7].

**A0** \(\max \{\mu(U), \mu(V)\} \leq \mu_0 \in \mathbb{R}_{++}\).  

**A1** \(\|\sum_{i \in \mathbb{N}_N^r} u_i v_i^H\|_\infty \leq \mu_1 \sqrt{\frac{r}{N_1 N_2}}, \quad \mu_1 \in \mathbb{R}_{++}\).  

If a matrix \(M\) obeys the assumptions A0 and A1 with parameters \(\mu_0\) and \(\mu_1\), we will say that it obeys the incoherence property or, equivalently, that it is incoherent with parameters \(\mu_0\) and \(\mu_1\).

In [6], the respective authors implicitly replace subspace coherence with a closely related quantity, which we will refer to as strong subspace coherence, enabling them to prove tighter -in fact, almost optimal- bounds on the number of observations required in order to achieve exact reconstruction of \(M\) by solving (1) and whose definition is presented below.

**Definition 2. (Strong Subspace Coherence)** Consider the hypotheses of Definition 1. Then, the strong coherence of \(U\) with respect to the standard basis \(\{e_i\}_{i \in \mathbb{N}_N}\) is defined as

\[
\mu_s(U) \triangleq \sup_{(i,j) \in \mathbb{N}_N^r \times \mathbb{N}_N^r} \left| \frac{N}{r} \langle e_i, P_U e_j \rangle - 1_{i=j} \right|.
\]

(3)

Using the definition stated above, the authors in [6] essentially replace A0 by the following assumption, concerning the singular vectors of \(M\).

**A2** \(\max \{\mu_s(U), \mu_s(V)\} \leq \frac{\mu_0^*}{\sqrt{r}}, \quad \mu_0^* \in \mathbb{R}_{++}\).  

If \(M\) obeys the assumptions A1 and A2 with a parameter \(\mu \geq \max \{\mu_0^*, \mu_1\}\), we will say it obeys the strong incoherence property or, equivalently, that it is strongly incoherent with parameter \(\mu\). In this case, powerful exact reconstruction theorems hold (e.g., see Theorems 1.1 and 1.2 in [6]).

Further, it is easy to show that, in fact, incoherence implies strong incoherence [6], as the following lemma asserts.

**Lemma 1. (Incoherence Model Equivalence [8])** If a matrix \(M \in \mathbb{C}^{N_1 \times N_2}\) of rank \(r\) is incoherent with parameters \(\mu_0\) and \(\mu_1\), it is also strongly incoherent with parameter \(\mu \leq \mu_0 \sqrt{r}\).

Of course, in any realistic setting, the available matrix observations will be corrupted by noise. However, as the authors in [9] aptly explain, “when perfect noiseless recovery occurs, then matrix completion is stable vis a vis perturbations”. Consequently, in order to guarantee satisfactory performance for matrix completion, even in the noisy case, it suffices to study the conditions under which the coherence of \(M\) (through Lemma 1) will be as low as possible.

**B. Matrix Completion in Colocated MIMO Radar**

We consider the respective problem formulation proposed in [4] (see Subsection A of Section II in [4]), where the matrix to be completed at the fusion center of the receiver, \(\mathcal{P}(Y)\), obeys the special observation model

\[
Y \triangleq \Delta + Z \in \mathbb{C}^{M_r \times M_t},
\]

(4)

where \(M_r\) and \(M_t\) denote the numbers of reception and transmission antennas, respectively, \(Z\) is an interference/observation noise matrix and

\[
\Delta \triangleq X, DX^T,
\]

(5)

where the Vandermonde matrix \(X \in \mathbb{C}^{M_r \times K}\) (respectively for \(X_r \in \mathbb{C}^{M_r \times K}\)) is defined by the generating vector

\[
\Gamma \triangleq \left[\gamma_0, \gamma_1, \ldots, \gamma_{K-1}\right]^T \in \mathbb{C}^{K \times 1}, \quad \text{with}
\]

\[
\gamma_k \triangleq e^{2\pi i k \alpha_k}, \quad \alpha_k \triangleq \frac{d_r \sin(\theta_k)}{\lambda}, \quad k \in \mathbb{N}_{K-1},
\]

(6)

and the set \(\{\theta_k\}_{k \in \mathbb{N}_{K-1}}\) containing the target angles and \(d_r\) and \(\lambda\) denoting the (receiver) array antenna spacing and carrier wavelength utilized for the communication, respectively, and where \(D \in \mathbb{C}^{R \times K}\) is an arbitrary non-zero - diagonal matrix defined as

\[
D \triangleq \text{diag}(z_k, z_2, \ldots, z_K)\), \quad \text{with}
\]

\[
\rho_i \triangleq \exp \left( j \frac{4\pi}{\lambda} \vartheta_i (q-1) T_{PR} \right), \quad i \in \mathbb{N}_{K-1},
\]

(7)

with the sets \(\{z_k\}_{k \in \mathbb{N}_{K-1}}\) and \(\{\vartheta_i\}_{i \in \mathbb{N}_{K-1}}\) containing the target reflection coefficients and target speeds, respectively, and \(q\) and \(T_{PR}\) denoting the pulse index and the pulse repetition interval, respectively.

**III. RECOVERABILITY OF \(\Delta\) AND MC PERFORMANCE BOUNDS**

In this section, we present our main results. In short, we prove:

- Asymptotic and approximate optimality of the coherence of \(\Delta\) with respect to the number of transmission/reception antennas and

- Near optimal recoverability of \(\Delta\) via matrix completion.  

For detailed proofs of our results presented below, the reader is referred to [8].
A. Coherence of $\Delta$

**Theorem 1. (Coherence for ULAs)** Consider a Uniform Linear Array (ULA) transmitter - receiver pair and assume that the set of target angles $\{\theta_k\}_{k \in \mathbb{N}_{K-1}}$ consists of almost surely distinct members. Then, for any fixed $M_t$ and $M_r$, as long as

$$K \leq \min_{i \in \{t, r\}} \left\{ \frac{M_i}{\sqrt{\beta_{\xi_k}(M_i)}} \right\},$$  

(10)

the associated matrix $\Delta$ obeys the assumptions A0 and A1 with

$$\mu_0 \equiv \max_{i \in \{t, r\}} \left\{ \frac{M_i}{M_i - (K - 1) \sqrt{\beta_{\xi_k}(M_i)}} \right\} \quad \text{and} \quad \mu_1 \equiv \max_{i \in \{t, r\}} \left\{ \frac{M_i \sqrt{K}}{M_i - (K - 1) \sqrt{\beta_{\xi_k}(M_i)}} \right\},$$  

(11)-(12)

with probability 1. In the above, $\beta_{\xi_k}(M_k) \equiv \sup_{x \in \left[\xi_k, \xi_k + \frac{1}{2}\right]} \frac{\sin^2(\pi M_k x)}{\sin^2(\pi x)}$, for $k \in \{t, r\}$ and

$$\xi_k \equiv \min_{i \neq j} \left\{ g \left( \frac{d_k}{\lambda} |\sin(\theta_i) - \sin(\theta_j)| \right) \right\}$$  

(13)-(14)

Further, if $\xi \equiv \min \{\xi_t, \xi_r\} \neq 0$, then, for any fixed $K$, as long as

$$\min_{i \in \{t, r\}} M_i \geq K \sqrt{\frac{2}{\beta_{\xi_k}}} = \mathcal{O}(K),$$  

(15)

where

$$\beta_{\xi_k} \equiv \sup_{x \in \left[\xi_k, \xi_k + \frac{1}{2}\right]} \frac{\sin^2(\pi M_k x)}{\sin^2(\pi x)} = \sup_{x \in \left[\xi_k, \xi_k + \frac{1}{2}\right]} \frac{\sin^2(\pi M_k x)}{\sin^2(\pi x)}.$$  

(16)-(17)

(11) and (12) hold replacing both $\beta_{\xi_k}(M_t)$ and $\beta_{\xi_k}(M_r)$ by the constant $\beta_{\xi}$ (that is, independent of both $M_t$ and $M_r$). Additionally, in the limit we respect to $M_t$ and $M_r$, we have

$$\mu(V) \equiv \mu(U) \equiv 1,$$  

(18)

that is, the coherence of $\Delta$ is asymptotically optimal. Finally, if $\xi$ is safely bounded away from zero, then, for sufficiently large $M_t$ and $M_r$, it is true that

$$\mu(V) \approx \mu(U) \approx 1.$$  

(19)

that is, the smallest possible coherence for $\Delta$ can be approximately attained even for finite values of $M_t$ and $M_r$.

In colocated MIMO Radar systems, due to the need for unambiguous angle estimation (target detection), it is very common to assume that $\theta_i \in [-\pi/2, \pi/2], \forall i \in \mathbb{N}_{K-1}$. Also, especially for the case where ULAs are employed for transmission and reception, another common assumption is to choose $d_r \equiv d_t = \lambda/2$. Under this setting, the following lemma provides a simple sufficient condition, which guarantees that the asymptotics of Theorem 1 hold true and, as a result, that for a sufficiently large number of transmission/reception antennas, the coherence of $\Delta$ will be small.

**Lemma 2. (ULA Pairs Coherence Control)** Consider the hypotheses and definitions of Theorem 1. Set $d_r \equiv d_t = \lambda/2$. If additionally

$$\vartheta = \left\{(\theta, \theta) \in \mathcal{A}, \quad m \leq \frac{\pi}{2}, \frac{\pi}{2} \right\}$$  

and the asymptotics of Theorem 1 always hold true. In particular, the higher the value of $\eta$, the higher the value of $\xi$ and the lower the coherence of $\Delta$.

B. Near Optimal Recoverability of $\Delta$

Using Lemma 1, we can now directly combine Theorem 1 with Theorems 1.1 and 1.2 of [6], producing (asymptotic) bounds regarding the number of observations required for the exact reconstruction of the matrix $\Delta$ by solving the convex program (1). Specifically, we present the following interesting result.

**Theorem 2. (Near Optimal Recoverability for ULA Pairs)** Consider the hypotheses of Theorem 1 and assume that $\xi$ is safely bounded away from zero. Also, set $M \equiv \max \{M_r, M_t\}$. Suppose we observe $m$ entries of $\Delta$ with locations sampled uniformly at random. Then, for $M_t$ and $M_r$ sufficiently large satisfying

$$\min_{i \in \{t, r\}} M_i \geq K \sqrt{\frac{2}{\beta_{\xi}}}$$  

(20)-(23)

for a fixed number of targets $K$, $\Delta$ is strongly incoherent with parameter

$$\mu \leq \mu_0 \sqrt{K} \approx \sqrt{K},$$  

(24)

and there exist positive numerical constants $C_1$ and $C_2$ such that if

$$m \geq C_1 K^4 M \log^2 M = \mathcal{O} \left( M \log^2 M \right),$$  

(25)

$$m \geq C_2 K^2 M \log^2 M = \mathcal{O} \left( M \log^2 M \right),$$  

(26)

the minimizer to the program (1) is unique and equal to $\Delta$ with probability at least $1 - M^{-3}$.

Roughly speaking, Theorem 2 implies that as long as ULAs are concerned, then, for a sufficiently large number of transmission and reception antennas and for a fixed and relatively small number of targets, matrix completion is exact.
if the number of observations is at least of an order of
\( M \) \( \text{polylog}(M) \), that is, matrix completion is exact for a
slightly larger number of observations than the information
theoretic limit (see [6] for details).

IV. DISCUSSION AND SIMULATIONS

In this section, we present and discuss some simulations,
validating our main results presented above.

For simplicity, we consider a MIMO radar system equipped
with identical ULAs for transmission and reception, with \( d_r \equiv d_t = \lambda/2 \) and \( M_r \equiv M_t \equiv M \). We also consider \( K = 4 \)
targets in the far field, with angles independently distributed
in \( \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \).

In this synthetic example, in order to demonstrate the
validity of Theorem 1, we assume that the target angles are
known apriori. By Lemma 2, we know that the condition \( \xi \neq 0 \)
will always hold and consequently the asymptotics of Theorem
1 must also hold true. Of course, \( \xi \) can be computed either
using (14) or (22) (for some sufficiently chosen value of \( \eta \)),
with the latter producing a worst-case bound regarding the
coherence of the associated matrix \( \mathbf{\Delta} \). Also, in this example,
we obviously have \( \mu(U) = \mu(V) \).

Using (14) for the computation of \( \xi \), Fig. 1 shows the
behavior of both the coherence of \( \mathbf{\Delta} \) and its bound \( \mu_0 \),
which results by directly applying Theorem 1, as a function of
the number of transmission/reception antennas \( M \). We observe
that, clearly, the proposed bound is tight and it tracks the
convergence of the coherence of \( \mathbf{\Delta} \) to unity very accurately.
We should mention here that for very small angle differences,
the bound becomes somewhat looser. However, our numerical
simulations have shown that our bound constitutes a very
reasonable coherence estimate in all cases.

Apparently, in any realistic situation, the actual values of the
target angles are unknown. Assuming generically that each of
the angle differences belongs to a set given by (21) for some
\( \eta \in \left( 0, \frac{\pi}{2} \right) \), \( -\eta \) depends on the specific radar application,–
we can invoke Lemma 2 in order to bound the coherence of
\( \mathbf{\Delta} \) in this more general case. Fig. 2 depicts a number of bounds
produced by Lemma 2 for various values of \( \eta \), as functions
of the number of antennas \( M \). One can observe that, as the
value of \( \eta \) increases, the respective coherence bound converges
much faster to unity, therefore increasing our confidence that
the performance of matrix completion will be satisfactory for a
relatively smaller number of transmission/reception antennas.

V. CONCLUSION

In this paper, we have presented novel results regarding
the recoverability of the data matrix in ULA colocated MC
based MIMO radar systems. We showed that the data matrix
is indeed recoverable from a minimal number of observations,
as long as the number of transmission and reception antennas
is sufficiently large and under common assumptions on the
DOAs of the targets. Consequently, the matrix completion
approach for reducing the sampling requirements in colocated
MIMO radar is indeed theoretically robust and also appealing
for practical consideration in real world applications.

REFERENCES

[1] E. Fishler, A. Haimovich, R. Blum, L. Cimini, D. Chizhik, and R. Valen-
zuela, “Mimo radar: An idea whose time has come,” in Proc. of the IEEE
Int. Conf. on Radar, Philadelphia, Pa., 2004.
processing,” in Proc. of 37th International Conference on Acoustics,
mimo radar via matrix completion,” in Proc. of 38th International
Conference on Acoustics, Speech, and Signal Processing (ICASSP 2013),
May 26 - 31 2013.
approach based on matrix completion,” IEEE Trans. on Aerospace
and Electronic Systems, under review in 2013.
matrix completion,” IEEE Trans. on Information Theory, vol. 56, no. 5,
mimo radar: Recoverability, bounds and theoretical guarantees,” IEEE
Trans. on Signal Processing, under review in 2013.